A (linear) logical view of (linear) type isomorphisms

Roberto Di Cosmo

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Abstract

1 Introduction and Survey

- Cattivo cambio di $^{\perp}$, $\wp$ per $\multimap$: si perde la simmetria e cose semplici come associativita e commutativita diventano curry e swap e le regole per le unita si complicano enormemente (l'identita' per il tensore diventa un pasticcio da gestire con $\multimap$: $1 \multimap A = A$ é ben piu complicato che $\perp \wp A = A$. Voglio dire, mentre 1 e' l'identita destra e sinistra per il par, e' solo sinistra per $\multimap$. Inoltre, é chiaro perché $A \multimap 1$, che é $A^{\perp} \wp 1$, non fa 1! Da studiare la relazione con la linearità qui.
  Tuttavia, ci sono casi in cui la freccia lineare e' comoda per catturare per esempio la composizione de funzioni o strategie in semantica dei giochi.
- vera natura dei moltiplicativi
- legami con algebra dinamica
- problema delle unita

2 Linear isomorphisms of types

We focus in this paper on linear isomorphisms of types in multiplicative linear logic.

discutere il legame tra isomorphismi con $A^{\perp}\parr B$ e $A^{\perp}, B$. (ma no, é semplicemente la lettura normale degli isos logici, ma va detto)

Definition 2.1 (Linear isomorphism) *Two formulae A and B are* isomorphic *iff*

- *A and B are linearly equivalent, i.e. $\vdash A^{\perp}, B$ and $\vdash B^{\perp}, A$*
- *when we compose the proofs of $\vdash A^{\perp}, B$ and $\vdash B^{\perp}, A$ using a cut rule to obtain a proof of $\vdash A^{\perp}, A$ (resp. $\vdash B^{\perp}, B$)), after cut elimination, we obtain a proof reduced to the axiom $\vdash A^{\perp}, A$ (resp $\vdash B^{\perp}, B$)*

3 Simple proof nets

Proposition 3.1 (balanced hypothesis) *Whenever $\vdash_{MLL} A$, the number of negative and positive occurrences of an atom p in A are the same.*

Proof.
□

Definition 3.2 (simple nets) *A proof net is simple if it contains only atomic axiom links.*

Definition 3.3 (η-expansion of proof nets) *For any (possibly not simple) proof net S, there is a simple proof net with the same conclusions, obtained via a full $\eta - expansion$ of non atomic axiom links, which we call $\eta(S)$.*

This shows that, w.l.o.g., we can restrict our attention to simple nets.

Definition 3.4 (tree of a formula, identity simple net) *A cut-free simple proof net S proving A is actually composed of the* tree *of A, (named $T(A)$), and a set of axiom links over atomic formulae. We call* identity simple net *of A the simple cut-free proof net obtained by a full η-expansion of the (generally not simple) net $\overline{A \qquad A^{\perp}}$. This net is made up of $T(A)$, de $T(A^{\perp})$ and a set of axiom links that connect atoms in $T(A)$ with atoms in $T(A^{\perp})$.*
Notice that, in simple nets, the identity axiom for A is interpreted by the identity simple net of A.

Definition 3.5 (non-ambiguous formulae) *We say that a formula A is non-ambiguous if each atom in A occurs at most once positive and at most once negative. For example, is not non-ambiguous, $A\otimes B$ et $A\otimes A^{\perp}$ are non-ambiguous, while $A\otimes A$ is not.*

In what follows, we will focus only on non-ambiguous formulae. **??????SERVE?????**

A first remark, which is important for a simple treatment of linear isomorphisms, is that we can focus, w.l.o.g., on witnesses of isomorphisms which are simple proof nets. **credo di no**

Lemma 3.6 (Simple vs. non-simple nets) *If a (non-simple) net S reduces via cut-elimination to S', then the simple net $\eta(S)$ reduces to $\eta(S')$.*

Theorem 3.7 (Reduction to simple proof nets) *Two formulae A and B are* isomorphic *iff there are two simple nets S with conclusions $A^{\perp}, B$ and S' with conclusions $B^{\perp}, A$ that when composed using a cut rule over B (resp. A) yield after cut elimination the identity simple net of A (resp. B).*

Proof. The only if direction is trivial, since a proof net represents a proof and cut elimination in proof nets correspond to cut elimination over proofs.
For the if direction, take the two proofs giving the isomorphism and build the associated proof nets S and S'. These nets have as conclusions $A^{\perp}, A$ (resp. $B^{\perp}, B$), and we know that after composing them via cut over B (resp. A) and performing cut elimination, one obtains the axiom net of A (resp. B) after composition. Now take the full η-expansions of S and S' as the required simple nets: by lemma 3.6, they reduce by composition over B (resp. A) to the identity simple net of A (resp. B). $\square$

We will show now that if two non-ambiguous formulae are isomorphic then the isomorphism can be given by means of proof nets whose structure is particularly simple.

Definition 3.8 (biparte simple proof-nets) *A cut-free simple proof net is* biparte *if it has exactly two conclusions A and B, and it consists of $T(A)$, $T(B)$ and a set of axiom links connecting atoms of A to atoms of B, but not atoms of A between them or atoms of B between them.*

Lemma 3.9 (cuts and trees) *Let S be a simple net (not a proof net) without conclusions built out of just $T(A)$ and $T(A^{\perp})$, with no axiom link, and the cut $\underline{A \quad A^{\perp}}$. Then cut-elimination on S yields as a result just a set of atomic cut links $\underline{p_i \quad p_i^{\perp}}$ between atoms of A and atoms of $A^{\perp}$.*

Proof. This is a simple induction on the size of the net. $\square$

We are now able to state our main results.

Theorem 3.10 (correctness) *Let S be a bipartite simple proof net over $A^\perp$ and B, and S' a bipartite simple proof net over $B^\perp$ and A. Then their composition by cut over B reduces to the identity simple net of A (resp. their composition by cut over A reduces to the identity simple net over B).*

Definition 3.11 (formal inverse) *For each bipartite simple proof net S over $A^\perp$ and B, we can define the formal inverse S^{-1} of S by taking in it just the dual of each formula of S.*

Remark 3.12 *This statement is false if the net is not bipartite: $((A \multimap B) \otimes A) \multimap B$ is provable, but not $B \multimap ((A \multimap B) \otimes A)$*

Theorem 3.13 (completeness) *Let S be a cut-free simple proof net with conclusions $A^\perp$ abd B, and S' be a cut-free proof net with conclusions $B^\perp$ et A. if their composition by cut gives respectively the identity simple net of A and B, then S and S' are biparte.*

These two theorems have the following fundamental consequence.

Corollary 3.14 *Two linear formulae A and B are isomorphic iff and only if there exist a simple bipartite proof net having conclusions $A^\perp, B$.*

4 Completeness for isomorphisms in *MLL*

Theorem 4.1 (Isos soundness) *If $AC(\otimes, \mathbin{⅋}) \vdash A = B$, then A and B are linearly isomorphic*

Proof. By exhibiting the simple nets for the axioms and showing context closure. □

Theorem 4.2 (Isos completeness) *If A and B are linearly isomorphic, then $AC(\otimes, \mathbin{⅋}) \vdash A = B$.*

Proof. A simple induction on the number of tensor nodes in the associated bipartite simple proof net, using in an essential way the correctness criterion.
If there are no tensor nodes, then the net is necessarily reduced to an atomic axiom link (otherwise, it is easy to exhibit a disconnected correctness graph: take two

different maximal paths: they must be distinct in at least two nodes, which are necessarily ⅋ nodes, and then we are done).
Otherwise, at each step, remove all dangling ⅋ nodes and consider then the splitting tensor node (that must exist due to girard's correctness criterion). Removing this tensor node yields, due to the correctness criterion, two disconnected proof nets, which are still simple and bipartite, since we did not modify the axiom links. On these two nets we can apply the induction hypothesis, and then conclude using associativity and commutativity of ⅋ and $\otimes$. □

5 Dynamic Algebra, GOI and invertibility

Parlare un po' dell'invertibilità dei cammini nell'algebra: matrici 2x2 simil-permutazione con decomposizioni dell'identita' come componenti. Vedere il mail che mi ero spedito con i conti fatti in Maple.

6 Handling the units

We have shown above soundness and completeness result for the theory of isomorphisms given in the introduction w.r.t. *provable* isomorphisms in *MLL*. This essentially corresponds to isomorphisms in all *-autonomous categories, which is a superset of all Symmetric Monoidal Closed Categories (SMCC's) *without units*. Nevertheless, if we want to get an interesting result also in terms of models, and handle then also SMCC's in their full form, we need to be able to add units to our treatment.

check: units in def di *-autonomous?

6.1 Expansions of axioms with units: identity simple nets revisited

In the presence of the units and especially $\perp$, proof nets in general get more involved, as $\perp$ forces the introduction of the notion of *box* for which we refer the interested reader to [**?**], where a detailed explanation is presented.

For our purpose, it will suffice to recall here the proof-net formation rules for the units:

$$\frac{\boxed{\begin{array}{c}\vdots\\ \Gamma\end{array}}}{\bot\Gamma}$$

Now, the expansion of an axiom can contain boxes, if the axiom formula involves units; for example, the axiom $\vdash (A\otimes)^{\bot}, (A\otimes)$ gets fully η-expanded into:

6.2 Reduction of isomorhpisms to simple nets with units

6.2.1 Cuts with units: a missing case

In girard's original paper, the only cut condition involving explicitly considered was:

$$\frac{bordel}{CUT} \qquad \text{reducing to} \qquad \begin{array}{c}\vdots\\ \Gamma\end{array}$$

There is a simple case that is also possible, though,

$$\frac{bordeletbordel}{CUT} \qquad \text{reducing to} \qquad \frac{\boxed{\begin{array}{c}\vdots\\ \Gamma\end{array}}}{\bot\Gamma}$$

6.3 Completeness with units

7 Linear isomorphisms in linear lambda calculus

this section should already have been handled by Soloviev or somebody else to show completeness for isomorphisms of SMCC w.r.t. linear lambda calculus.

We can try to do this through a coding into *MLL* with units.

Theorem 7.1 (Soundness for linear lambda calculus isomorphisms) *If A and B are arrow types (with unit) isomorphic in the linear lambda calculus, then the translations of A and B are isomorphic in MLL (with units).*

Proof. Take the full η-expansion of the simple translations of the lambda terms giving the isomorphism: they provide us with the invertible proofs in *MLL* (with units). □

check: nonsono cut-free le traduzioni

Theorem 7.2 (Completeness for linear lambda calculus isomorphisms) *If A and B are arrow types (with unit) whose translations are isomorphic in MLL (with units), then A and B are isomorphic in the linear lambda calculus.*

Proof. The linear lambda terms can be built out of the proof nets giving the isomorphism, by linearisation. □

8 Conclusions