

A confluent reduction for the λ -calculus with surjective pairing and terminal object

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Abstract

We exhibit confluent and effectively weakly normalizing (thus decidable) rewriting systems for the full equational theory underlying cartesian closed categories, and for polymorphic extensions of it. The λ -calculus extended with surjective pairing has been well-studied in the last two decades. It is not confluent in the untyped case, and confluent in the typed case. But to the best of our knowledge the present work is the first treatment of the lambda calculus extended with surjective pairing *and* terminal object via a *confluent* rewriting system, and is the first solution to the decidability problem of the full equational theory of Cartesian Closed Categories extended with *polymorphic types*. Our approach yields conservativity results as well. In separate papers we apply our results to the study of provable type isomorphisms, and to the decidability of equality in a typed λ -calculus with subtyping.

Résumé

Nous présentons des systèmes de réécriture confluents et effectivement faiblement normalisants pour la théorie équationnelle complète des catégories cartésiennes fermées, et pour des extensions polymorphes de cette théorie. Le λ -calcul avec paires surjectives a été bien étudié ces vingt dernières années. Il est non confluent dans le cas non-typé, et confluent dans le cas typé. A notre connaissance, ce travail propose le premier traitement du λ -calcul avec paires surjectives *et* objet terminal à l'aide d'un système de réécriture *confluent*, et la première solution du problème de la décidabilité de la théorie équationnelle complète des CCC étendue avec du *polymorphisme*. Notre approche conduit aussi à des résultats de conservativité. Dans d'autres articles, nous appliquons nos résultats à l'étude des isomorphismes de types, et de la décidabilité de l'égalité dans un λ -calcul étendu par du sous-typage.

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1 Introduction

Since 1972 there has been some interest in the properties of λ -calculus extended with products and *surjective pairing* (SP), that lead to J.W. Klop's discovery [Klo80] that for pure lambda calculus this extension, that we will note $\lambda^1\beta\eta\pi$, fails to maintain confluence¹, while it remains unproblematic [Pot81] for the typed calculus. Due to the connection with Cartesian Closed Categories (ccc's), another extension of the typed calculus has been considered: $\lambda^1\beta\eta\pi*$, that is $\lambda^1\beta\eta\pi$ with *terminal object*. This calculus is relevant for the decision problem of the equational theory of ccc's and for the coherence problem for the same categories, which are discussed in [LS86] and [Min] respectively. Neither of these works provides a truly confluent reduction system for the full calculus: the former takes advantage of type isomorphisms to "eliminate" the terminal object and reduces the full decision problem to the decision problem for $\lambda^1\beta\eta\pi$ only, the latter gives a system that is Church-Rosser only up to a congruence.

More recent is the interest in $\lambda^1\beta\eta*$, the calculus extended with a terminal object only and no products, that arose in the study of the theory of object oriented programming. In the framework of inheritance, the terminal type \mathbf{T} has an additional flavour: it is a maximum type. Type inclusion is *not* invariant under isomorphisms, so that, say $A \times T$ is a type greater than $A \times A'$ for any A' , while the same is not true of A^2 .

Thus the method of solving word problems by first getting rid of the terminal object as in [LS86] is of no use in the syntactic theory of λ -calculi with subtyping. We rather need a confluent system for the full type system, terminal (or maximum) type included.

In this paper we exhibit confluent and effectively weakly normalizing (thus decidable) rewriting systems for the full equational theory underlying cartesian closed categories, and for polymorphic extensions of it. To the best of our knowledge, this work provides the first solution to the decidability problem of the full equational theory of Cartesian Closed Categories extended with polymorphic types. Moreover we can take profit of confluence to get conservativity results in addition to decision results. Such conservativity results are needed in the study of provable type isomorphisms.

The results are applied in two companion papers:

- [CG90] establishes a decidability result in the paradigmatic language F_{\leq} , a variant of second-order λ -calculus with a maximum type and bounded quantification: the equational theory considered consists of β , η (first and second-order) and the terminal type rule. We show the confluence of our system via a translation to the polymorphic λ -calculus with a terminal

¹See [Bar84], p. 403-409 for a short history and references.

²Recently, L. Cardelli has proposed the following nice and simple exploitation of \mathbf{T} as a maximum type: consider the well-known inheritance $[\text{age};\text{sex}]$ less than $[\text{age}]$; encode $[\text{age}]$ as $\text{age} \times T$ and $[\text{age};\text{sex}]$ as $\text{age} \times (\text{sex} \times T)$. Then the desired subtyping obviously holds componentwise, by reflexivity and maximality, respectively.

type (what is called hereafter $\lambda^2\beta\eta\pi^*$), and by using a general criterion allowing to transfer confluence in $\lambda^2\beta\eta\pi^*$ back to our source system.

- [BDCL90] and [DC91] give an equational characterization of all type isomorphisms which are provable in the typed λ -calculus (respectively second order λ -calculus) with pairs and terminal object (what is called hereafter $\lambda^1\beta\eta\pi^*$, respectively $\lambda^2\beta\eta\pi^*$). It turns out that this characterization can be given quite easily if we are able to determine the structure of *invertible* terms, i.e. terms that possess an inverse w.r.t. the usual operation $\lambda x.\lambda y.\lambda z.(x(yz))$ of composition. The conservativity of equality in the extended calculus over the calculus without products and terminal objects allows us to reduce the problem to the invertibility in the simply typed (respectively second-order) λ -calculus³.

Technically, we had to navigate between several pitfalls before we arrived to our solution. We survey the main steps of this eventually safe trip in the next section. Sections 3 and 4 are devoted to confluence and weak normalization respectively. In section 5 we state the decidability and conservativity results that follow quite obviously from confluence and weak normalization, and we put our work in perspective with the other approaches to decidability of the same theories that we are aware of. Section 6 is a brief conclusion.

2 Survey

After defining precisely the calculi we focus on, we use the Knuth-Bendix procedure by hand to obtain locally confluent rewriting systems. We then shortly hint at a severe technical difficulty in adapting the standard strong normalization proofs which use the so called reducibility method. They can be adapted to a subsystem only. From the confluence of this subsystem we get confluence of almost the whole system by a general criterion presenting an interest of its own. At this stage, only the second-order β -rule is left out, and it can be finally added with the help of Hindley-Rosen's Lemma. As for weak normalization, the ingredients developed for confluence give it for free for first-order systems, while for the second order systems another splitting in subsystems, and another adaptation of the standard strong normalization proofs are needed.

We give now the full definition of the calculus $\lambda^2\beta\eta\pi^*$, the most complex of the four we consider.

2.1 The calculus $\lambda^2\beta\eta\pi^*$

Definition 2.1 *$\lambda^2\beta\eta\pi^*$ is the extension of second order lambda calculus defined*

³Ultimately the problem is reduced to the invertibility in the untyped λ -calculus (see [Bar84], section 21.2), where invertible terms have a simple (but not easy to prove!) syntactic characterization due originally to Mariangiola Dezani [Dez76].

as follows:

- Types are defined by the following grammar:

$$Type ::= At \mid Var \mid Type \rightarrow Type \mid Type \times Type \mid \forall X. Type$$
 where At are countably many atomic types and Var countably many type variables
- Terms ($M:A$ will stand for M is a term of type A)
 - the set of terms contains countably many variables x, y, \dots of each type
 - $*$: \mathbf{T}
 - if x is a variable of type A and $M:B$, then $\lambda x.M:A \rightarrow B$
 - if $M:A \rightarrow B$ and $N:A$, then $(M N):B$
 - if $M:A$ and $N:B$ then $\langle M, N \rangle:A \times B$
 - if $M:A \times B$ then $p_1M:A$ and $p_2M:B$
 - if $M:A$ and X is a type variable not free in the type of any free variable of M , then $\Lambda X.M:\forall X.A$
 - if $M:\forall X.A$ and B is a type, then $M[B]:A[B/X]$.

Notice that pairing and projections are new *term formation rules* and not constants added to the language.

- Equality

$$\begin{array}{ll}
 (\beta) & (\lambda x.M)N = M[N/x] & (\eta) & \lambda x.Mx = M \text{ if } x \notin FV(M) \\
 (\pi) & p_i \langle M_1, M_2 \rangle = M_i & (SP) & \langle p_1M, p_2M \rangle = M \\
 & & (top) & M = * \text{ if } M : T \\
 (\beta^2) & (\Lambda X.M)[A] = M[A/X] & (\eta^2) & \Lambda X.M[X] = M \text{ if } X \text{ is not free in } M
 \end{array}$$

We will note $=_{\beta^2\eta^2\pi*}$ the theory of equality generated by $\beta, \eta, \pi, SP, top, \beta^2$ and η^2 .

The other calculi we are interested in can be naturally defined as restrictions of $\lambda^2\beta\eta\pi*$: to obtain them we reduce the class of types and/or terms, and accordingly redefine the equality. The calculus $\lambda^2\beta\eta*$ is $\lambda^2\beta\eta\pi*$ without product types, pairing and projections. (Equality for $\lambda^2\beta\eta*$ will be noted $=_{\beta^2\eta^2*}$ and is generated by $\beta, \eta, top, \beta^2$ and η^2). The calculus $\lambda^1\beta\eta\pi*$ is $\lambda^2\beta\eta\pi*$ restricted to the first order. (Equality for $\lambda^1\beta\eta\pi*$ will be noted $=_{\beta\eta\pi*}$ and is generated by β, η, π, SP and top). The calculus $\lambda^1\beta\eta*$ is the restriction of $\lambda^1\beta\eta\pi*$ obtained by removing product types, pairing and projections. (Equality for $\lambda^1\beta\eta*$ will be noted $=_{\beta\eta*}$ and is generated by β, η and top).

2.2 Weakly confluent reduction

We will adopt the following

Notation 2.2 (*Reductions*) As usual, \rightarrow will denote one-step reduction, while $\rightarrow_=_$ is the reflexive closure of \rightarrow , and \rightarrow^* is the reflexive transitive closure of \rightarrow . If the system we consider is weakly normalizing, we will note $\rightarrow^* \mid$ the reduction to a normal form. Also, *WN* will stand for weakly normalizing, *SN* for strongly normalizing, *CR* for confluent (or Church-Rosser) and *WCR* for weakly (or locally) confluent.

The systems obtained by orienting the equalities of $=_{\beta^2\eta^2\pi^*}$ and its restrictions are far from being even weakly confluent, due to a bad interaction between the rule *top* on one side and the rules η and *SP* on the other⁴. The point is that all terms of type \mathbf{T} are identified (in particular, $x:\mathbf{T}$ and $*$ are identical), so that $\lambda x:\mathbf{T}.Mx$ and $\lambda x:\mathbf{T}.M*$ are “the same” term, and must give rise to the same reductions: since the first reduces to M , the second must reduce to M too. This fact actually shows up during the completion procedure. Let us consider the typical critical pairs that arise, say, for $\lambda^2\beta\eta\pi^*$: after the first “stage” we find the situation described in figure 1.

	M	M'	M''	New reduction from completion
<i>eta-like</i>	$\lambda x:\mathbf{T}.Mx$	M	$\lambda x:\mathbf{T}.M*$	$\lambda x:\mathbf{T}.M* \rightarrow M$ if $x \notin \text{FV}(M)$
	$\langle p_1M, p_2M \rangle$	M	$\langle p_1M, * \rangle$	$\langle p_1M, * \rangle \rightarrow M$ if $M:A \times T$
	$\langle p_1M, p_2M \rangle$	M	$\langle *, p_2M \rangle$	$\langle *, p_2M \rangle \rightarrow M$ if $M:T \times B$
<i>top-like</i>	$\lambda x:A.Mx$	M	$\lambda x:A.*$	$M \rightarrow \lambda x:A.*$ if $M:A \rightarrow T$
	$\Lambda X.M[X]$	M	$\Lambda X.*$	$M \rightarrow \Lambda X.*$ if $M:\forall X.T$

Figure 1: The critical pairs at the first stage of Knuth-Bendix completion.
(M' is reached via η or *SP*; M'' via *top*)

The additional rules generated by completion can be divided in two groups: rules that behave like η (*eta-like*) and rules that behave like *top* (*top-like*). The former mimic the behaviour of η and *SP* rules on terms that are known to be “the same terms as” η and *SP* redexes, as in the example we just considered above. The latter force to identify all the terms of type $A \rightarrow T$ and $\forall A.T$, and do pick up a canonical representative in the respective types. It turns out that a set of *eta-like* rules must be generated for each of all types isomorphic (in the categorical sense, see [BDCL90] and [DC91]) to \mathbf{T} . At stage n , the completion procedure on one side creates new rules to mimic η and *SP* on terms that are

⁴This observation seems to have been first made by A. Obtulowicz, cf. [LS86], exercise at page 88.

known to be “the same” as *eta-like* stage n-1 redexes, and on the other side it discovers new “same” terms, following the pattern:

- if A is known to be isomorphic to \mathbf{T} at stage n-1, then $B \rightarrow A$ and $\forall X.A$ are isomorphic to \mathbf{T} at stage n
- if A and B are known to be isomorphic to \mathbf{T} at stage n-2, then $A \times B$ is isomorphic to \mathbf{T} at stage n.

These correspond to the well known isomorphisms $T \times T \cong \mathbf{T}$, $A \rightarrow T \cong \mathbf{T}$ and $\forall X.T \cong \mathbf{T}$. (The isomorphism $T \times T \cong \mathbf{T}$ shows up only from the second stage on: consider the stage 1 *eta-like* redex $\langle *, p_2 M \rangle$, and suppose $M:T \times T$. Then we reach M by the *eta-like* reduction, and $\langle *, * \rangle$ by *top*.)

The following notation will allow us to present in a compact formalism the resulting weakly confluent reduction system.

Definition 2.3 *Terminal types and Canonical terms.*

1. $Iso(T)$ (the collection of types isomorphic to \mathbf{T}) is the set defined as follows:

- (a) $\mathbf{T} \in Iso(T)$
- (b) if $B \in Iso(T)$, then $A \rightarrow B \in Iso(T)$ for every type A
- (c) if $A \in Iso(T)$ and $B \in Iso(T)$, then $A \times B \in Iso(T)$
- (d) if $A \in Iso(T)$ and X is a type variable, then $\forall X.A \in Iso(T)$.

2. for each type $A \in Iso(T)$, the associated *canonical* representative $rep(A)$ is defined inductively as follows:

- (a) $rep(\mathbf{T})$ is $*$
- (b) $rep(A \rightarrow B)$ is $\lambda x:A.rep(B)$
- (c) $rep(A \times B)$ is $\langle rep(A), rep(B) \rangle$
- (d) $rep(\forall X.A)$ is $\Lambda X.rep(A)$.

Definition 2.4 $\xrightarrow{\beta^2 \eta^2 \pi^*}$ is the notion of reduction for $\lambda^2 \beta \eta \pi^*$ generated by orienting to the right the equalities β , η , π , SP , β^2 and η^2 in definition 2.1 and adding the following rewriting rules, coming from completion:

- (*gentop*) $M:A \xrightarrow{\beta^2 \eta^2 \pi^*} rep(A)$ if $M:A$ and $A \in Iso(T)$ and M is not already $rep(A)$
- (SP_{top}) $\langle rep(A), p_2 M \rangle \xrightarrow{\beta^2 \eta^2 \pi^*} M$ if $M:A \times B$
- (SP_{top}) $\langle p_1 M, rep(B) \rangle \xrightarrow{\beta^2 \eta^2 \pi^*} M$ if $M:A \times B$
- (η_{top}) $\lambda x:A.M rep(A) \xrightarrow{\beta^2 \eta^2 \pi^*} M$ if $A \in Iso(T)$ and $x \notin FV(M)$.

The notions of reduction for the simpler calculi can be defined as restrictions of $\xrightarrow{\beta^2\eta^2\pi^*}$. The notion of reduction for $\lambda^2\beta\eta*$, that we will note $\xrightarrow{\beta^2\eta^2*}$, is the reduction induced on $\lambda^2\beta\eta*$ by $\xrightarrow{\beta^2\eta^2\pi^*}$, that is to say $\xrightarrow{\beta^2\eta^2\pi^*}$ without π , SP , and SP_{top} , as these rules cannot apply to terms of $\lambda^2\beta\eta*$. For the same reason, the clauses for product types in Definition 2.3 will never be used, so that actually only a restricted version of $gentop$ is used in $\xrightarrow{\beta^2\eta^2*}$. We shall still use $gentop$ to name this restricted reduction, as the intended meaning will always be clear from the context.

Similarly, $\xrightarrow{\beta\eta\pi^*}$ and $\xrightarrow{\beta\eta*}$ are the reductions induced by $\xrightarrow{\beta^2\eta^2\pi^*}$ on $\lambda^1\beta\eta\pi*$ and $\lambda^1\beta\eta*$, with the appropriate restrictions of $gentop$.

It is now just a matter of an easy structural induction on terms to see that

Proposition 2.5 $\xrightarrow{\beta^2\eta^2\pi^*}$ is weakly confluent (*WCR*).

What about confluence then? We cannot use the standard Tait-Martin Lőf “parallel reduction” technique, as the non-linear rule SP may require more than one adjustment step, which cannot be parallelized. Specifically, suppose that M one step reduces to M' : then $\langle p_1M, p_2M \rangle$ reduces both to M and to $\langle p_1M', p_2M \rangle$. The local confluence diagram can be completed on one side in one step to M' , but on the other side one must go sequentially to $\langle p_1M', p_2M' \rangle$, where the lost SP redex is recreated, and then to M' : this is hardly parallel.

2.3 Investigating Strong Normalization

Another “obvious” approach to prove confluence is to attempt to show that these notions of reduction are strongly normalizing, as then one could apply the well known fact that $SN + WCR \Rightarrow CR$ ⁵. But here we face a serious problem: some of the new reduction rules, namely η_{top} and SP_{top} , prevent us from applying the usual reducibility techniques (see [GLT90], [LS86], [Tai67]), as we briefly sketch now.

All variations of the reducibility method require at some point to show a key statement like *if $v[u/x] \in RED_V$ for all $u \in RED_U$, then $\lambda x.v \in RED_{U \rightarrow V}$* , where RED_T is the set of reducible terms of type T , and where $RED_{U \rightarrow V}$ is the set of $s:U \rightarrow V$ s.t. $(s u) \in RED_V$ for all $u \in RED_U$.

An auxiliary property which is available is that, for $(s t):T$, one has $(s t) \in RED_T$ as soon as $s' \in RED_T$ for all s' which are one step reducts of $(s t)$.

So the proof of the key statement reduces to the proof that all one step reducts of $(\lambda x.v) u$ are reducible. Now, if v is $(v' *)$, then $(\lambda x.v) u$ can reduce to $(v' u)$ that is *not* $v[u/x]=v$, and we do not know if $(v' u)$ is reducible: this does not follow from any of the hypotheses we have at hand. A similar situation

⁵Known as *Newman's Lemma*. See [Bar84], pag. 58.

arises for SP_{top} when considering the corresponding lemma for pairs. (See the Remark A.14 in Section A).

But the difficulty suggests a solution. The above example is problematic only if u is different from $*$, and this cannot happen if we restrict our attention to terms in *gentop* normal form (*gentop* n.f.). For this to work out we have to check that *gentop* normal forms are stable under reduction. Otherwise the problem could dynamically show up later in the reduction. Unfortunately the β^2 rule does not preserve *gentop* normal forms:

Example 2.6 The second order term $(\Lambda X.\lambda x : X.\lambda y : (X \rightarrow A).yx)[T]$ is in *gentop* normal form, but its contractum $\lambda x:\mathbf{T}.\lambda y:T \rightarrow A.yx$ is not, and reduces to $\lambda x:\mathbf{T}.\lambda y:T \rightarrow A.y*$. \square

So we are forced to drop β^2 . Summarizing, so far we have hopes for confluence in the system which is restricted in two ways: we work only with *gentop* normal forms and we have abandoned β^2 . Indeed we show that this restricted system is strongly normalizing (Section A), thus confluent (the proof of local confluence is easily adapted to the subsystem). Then we lift the confluence result to the system $\xrightarrow{\beta\eta^2\pi^*}$, as we will note the notion of reduction induced on $\lambda^2\beta\eta\pi*$ by $\xrightarrow{\beta^2\eta^2\pi^*}$ less β^2 (see next subsection).

Finally we add up β^2 , which forms a confluent system that commutes with $\xrightarrow{\beta\eta^2\pi^*}$. So at last we can use Hindley-Rosen's Lemma⁶, and we get confluence for the full system $\xrightarrow{\beta^2\eta^2\pi^*}$.

2.4 A general criterion for confluence

To get the confluence of $\xrightarrow{\beta\eta^2\pi^*}$ from the confluence of its restriction to *gentop* normal forms, we apply the following general method. Recall that two reduction systems R and S are said to commute when, for every term P , if $P \xrightarrow{R} Q$ and $P \xrightarrow{S} Q'$, there exists a term Q'' such that $Q \xrightarrow{S} Q''$ and $Q' \xrightarrow{R} Q''$.

Lemma 2.7 *Let R be a reduction system that can be split in two subsystems $R1$ and $R2$ s.t.*

1. *$R1$ is weakly normalizing*
2. *the set of $R1$ normal forms is closed w.r.t $R2$ reductions*
3. *$R2$ is confluent on $R1$ normal forms*
4. *\xrightarrow{R} commutes with \rightarrow | $R1$ (see notation 2.2).*

⁶The Hindley-Rosen's Lemma asserts the obvious but useful property that two separately confluent, commuting subsystems form a confluent system.

Then R is confluent.

Proof. Under the hypothesis above, any two reductions \xrightarrow{R} starting from the same term can be completed to the commuting diagram shown in figure 2:

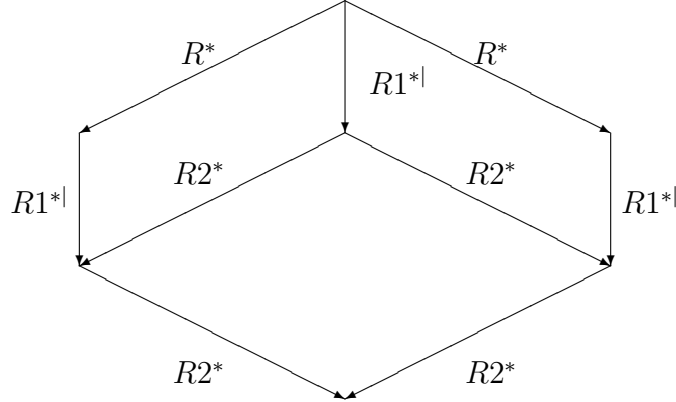


Figure 2: The factorization of confluence

- (1) ensures the existence of the R1 normal forms, hence we can build the central vertical arrow in the diagram ($R1^*|$ denotes reduction to some R1 n.f.).
- (4) ensures the existence and commutation of the upper inner rhombuses.
- (2) shows that the lower diagonal arrows in the upper rhombuses are made up of R2 reductions on R1 n.f.'s only, so that (3) guarantees the commutation of the lower inner rhombus.

Finally, the commutativity of the outermost rhombus follows from the commutativity of the inner rhombuses. \square

Notice that in the third condition, if the second condition holds, one may replace R2 by R. More interestingly, if we know that R1 is SN and confluent, then writing $R1(M)$ for the R1 normal form, we get the following alternative formulation of 4:

4' If $M \xrightarrow{R} M'$, then $R1(M) \xrightarrow{R} R1(M')$

where the second occurrence of R might as well be R2, and the first occurrence of R might as well be R^* . This form of the general criterion was already discovered and used by T. Hardin in her investigations of confluence properties of categorical combinators [Har89].

Our travel is close to the end. We shall take $\xrightarrow{\beta\eta^2\pi^*}$ as R, *gentop* as R1, $\xrightarrow{\beta\eta^2\pi^*}$ less *gentop* as R2 and prove the four conditions of the criterion. The confluence of R2 on R1 normal forms is proved by establishing WCR and SN.

3 Confluence

Let in the following R stand for one of $\xrightarrow{\beta\eta^2\pi^*}$, $\xrightarrow{\beta\eta^2}$, $\xrightarrow{\beta\eta\pi^*}$ or $\xrightarrow{\beta\eta^*}$, R1 be *gentop* and R2 be R less *gentop*. It will be intended that in the case of $\xrightarrow{\beta\eta\pi^*}$ and $\xrightarrow{\beta\eta^*}$, we consider only first order terms and types and hence only the corresponding restricted form of *gentop*, for which the following proofs hold almost unchanged.

We first introduce some notation.

Notation 3.1 We will note $(M)^T$ the *gentop* n.f. of a term M and $\xrightarrow{\text{gentop}}$ the reduction to *gentop* normal form $(M)^T$.

Lemma 3.2 The following equalities hold:

1. $(PQ)^T = (P)^T(Q)^T$ if $(PQ):A$ and $A \notin Iso(T)$
2. $(p_i P)^T = p_i(P)^T$ if $p_i P:A$ and $A \notin Iso(T)$
3. $(\lambda x.P)^T = \lambda x.(P)^T$
4. $(\langle P, Q \rangle)^T = \langle (P)^T, (Q)^T \rangle$
5. $(\Lambda X.P)^T = \Lambda X.(P)^T$
6. $(P[B])^T = (P)^T[B]$ if $P[B]:A \notin Iso(T)$.

Proof. We only check 3, and leave the rest to the reader. Let $\lambda x.P:A \rightarrow B$. If $A \rightarrow B \notin Iso(T)$, then the result is trivial, otherwise $(\lambda x.P)^T = rep(A \rightarrow B) = \lambda z.(P)^T$ for some fresh variable z . Since the *gentop* normal form of any term has no occurrence of variables in it (easily shown by induction), then $\lambda z.(P)^T$ is equal to $\lambda x.(P)^T$ by α -conversion. \square

Lemma 3.3 $\xrightarrow{\text{gentop}}$ is compatible with substitution, i.e.

$$(M[N/x])^T = (M)^T[(N)^T/x]$$

Proof. By an easy induction on the structure of M (see Table 1). Notice that the case $M:U$ and $U \in Iso(T)$ is trivial since in both cases the normal form is $rep(U)$, so in the table we consider only the case when the normal form of a compound term is the combination of the normal forms of its components.

\square

Lemma 3.4 *If $M \xrightarrow{R} M'$ then $(M)^T \xrightarrow{R} (M')^T$.*

Proof. We will proceed by induction on the structure of M . Notice that whenever M is a *gentop* redex, the claim holds trivially since the reductions we consider all preserve the type of the redex: so the type of M' is the same as that of M and their *gentop* normal forms are the same⁷. We shall thus assume that M is not a *gentop* redex. Furthermore, if the R reduction takes place in a proper subterm of M , the result follows easily by induction in each case (by Lemma 3.2), so we will not state it explicitly. We are left with the hypothesis that M is a redex which is not a *gentop* redex.

- M is a variable x . No reduction is possible, and the statement holds vacuously.
- M is an application. There is only one case:
 - M is $(\lambda x.P')Q$ and it β reduces to $P'[Q/x]$. Then $(M)^T = ((\lambda x.P')^T(Q)^T) = (\lambda x.(P')^T)(Q)^T$, and it β reduces to $(P')^T[(Q)^T/x]$, that is equal to $(P'[Q/x])^T$ by compatibility of \xrightarrow{gentop} with substitution (Lemma 3.3).
- M is an abstraction. There are two cases:
 - M is $\lambda x.(Px)$ and it η reduces to P . Then we have two possibilities for $(M)^T$ (notice that $(Px)^T = rep(V)$ is excluded as then M would be a *gentop* redex):
 - * $\lambda x.((P)^T x)$ which η reduces to $(P)^T$
 - * $\lambda x.((P)^T rep(U))$ which η_{top} reduces to $(P)^T$
 - M is $\lambda x.(Prep(U))$ and it η_{top} reduces to P . Then $(M)^T = \lambda x.((P)^T rep(U))$ that η_{top} reduces to $(P)^T$.
- M is a projection. The only case to consider is

⁷Remember that the contractum of a *gentop* redex depends only on the type of the redex, not on its structure.

– M is $p_i\langle P_1, P_2 \rangle$ and it π reduces to P_i . Then $(M)^T$ is $p_i(\langle P_1, P_2 \rangle)^T$, that is $p_i\langle (P_1)^T, (P_2)^T \rangle$, which π reduces to $(P_i)^T$.

• M is a pair. There are three cases:

– M is $\langle p_1P, p_2P \rangle$ and it SP reduces to P. By the previous lemma, we focus only on the following three possibilities for $(M)^T$:

* $\langle p_1(P)^T, p_2(P)^T \rangle$ that SP reduces to $(P)^T$

* $\langle p_1(P)^T, rep(V) \rangle$ that SP_{top} reduces to $(P)^T$

* $\langle rep(U), p_2(P)^T \rangle$ that SP_{top} reduces to $(P)^T$

– M is $\langle p_1P, rep(V) \rangle$ and it SP_{top} reduces to P.

Then $(M)^T$ is $\langle p_1(P)^T, rep(V) \rangle$ that SP_{top} reduces to $(P)^T$

– M is $\langle rep(U), p_2P \rangle$ and it SP_{top} reduces to P.

Then $(M)^T$ is $\langle rep(U), p_2(P)^T \rangle$ that SP_{top} reduces to $(P)^T$.

• M is an abstraction $\Lambda t.P$. There is only one case to consider, namely P is $P'[X]$ and reduces to P' via η^2 . We can assume $P'[X]$ not to be a *gentop* redex, as otherwise $M = \Lambda X.P'[X]$ would be a *gentop* redex too, while we already factored out the case $M:U \in Iso(T)$. By Lemma 3.2, $(M)^T = (\Lambda X.P'[X])^T = \Lambda X.(P'[X])^T = \Lambda X.(P')^T[X]$, that reduces via η^2 to $(P')^T$, as required.

Hence we have shown that $(M)^T \xrightarrow{R}_{=} (M')^T$. \square

Using the criterion for confluence, we will now show

Theorem 3.5 *R is confluent.*

Proof. We check the four hypotheses of lemma 2.7 for R split in R1 and R2 as above.

1. *gentop is a strongly normalizing confluent reduction system.*

Proof. Each *gentop* step strictly decreases the number of *gentop* redexes in the term it is applied to. Since it is also trivially WCR, Newman's Lemma applies and we get CR too. \square

2. *R2 reductions do not create new gentop redexes.*

Proof. By cases on the rule which is used. For all rules but β the result obviously follows from the fact that the reduct is a subterm of the redex. The case β is settled by noticing that, if M and N are in *gentop* n.f., then $M[N/x]$ is in *gentop* n.f. too. Indeed, this last property can be easily shown by induction on the structure of M .

If M is x or if it does not contain x free, then $M[N/x]$ is either M or N and the result follows from the hypothesis. We can also rule out the case where M is $\text{rep}(A)$, as then it has no free variables. So $M:A \notin \text{Iso}(T)$. If $M[N/x]$ contains a *gentop* redex P , then P cannot be $M[N/x]$, which has the same type as M , so P must be a proper subterm of $M[N/x]$. P cannot be a subterm of N either, or an unchanged subterm of M , as they are already in normal form, so it must be $M'[N/x]$ with M' a proper subterm of M containing a free occurrence of x . But M' is in *gentop* normal form as M is, hence, by induction hypothesis $M'[N/x]$ is not a *gentop* redex, so $M[N/x]$ is in *gentop* n.f. \square

3. *The systems $\xrightarrow{\beta\eta^2\pi^*}$, $\xrightarrow{\beta\eta^{2*}}$, $\xrightarrow{\beta\eta\pi^*}$ and $\xrightarrow{\beta\eta^*}$ are confluent over gentop normal forms.*

Proof. All the systems introduced so far are weakly confluent. We will prove in the appendix (theorem A.19, which follows closely the proof plan of [GLT90]), that $\xrightarrow{\beta\eta^2\pi^*}$ is strongly normalizing over *gentop* normal forms. This implies strong normalization (over *gentop* normal forms) for all the others subsystems of it. Hence they are confluent over *gentop* n.f.'s by Newman's Lemma. \square

v

4. *If $M \xrightarrow{R} M'$ then for any gentop n.f. N of M and N' of M' $N \xrightarrow{R} N'$.*

Proof. By Lemma 3.4 above and a simple diagram chase. \square

Remark 3.6 *Again, this statement holds for all the reduction systems we are considering, as we showed it for $\xrightarrow{\beta\eta^2\pi^*}$, and the statements for the others ones are particular cases of it.*

We can finally conclude, by lemma 2.7, that R is confluent. \square

We still have a gap to fill for the second-order systems, since we have left out β^2 . We shall prove CR for $\xrightarrow{\beta^2\eta^2\pi^*}$ and $\xrightarrow{\beta^2\eta^{2*}}$ by using Hindley-Rosen's Lemma.

Let $R1$ be the system $\xrightarrow{\beta\eta^2\pi^*}$ (or $\xrightarrow{\beta\eta^{2*}}$) and $R2$ be β^2 .

Lemma 3.7 β^2 is confluent.

Proof. The system consisting of β^2 alone satisfies the diamond property, hence is CR. \square

We just proved that R1 is CR (Theorem 3.5), so we are left to show that R1 commutes with R2, and the CR property will follow by Hindley-Rosen's Lemma.

Theorem 3.8 R1 and R2 commute with each other.

Proof. It suffices to prove that, if $M \xrightarrow{R1} M'$ and $M \xrightarrow{R2} N$, then there exist a term M'' s.t. $N \xrightarrow{R2} M''$ and $M' \xrightarrow{R1} M''$ (see Lemma 3.3.6 in [Bar84], pag. 65). The only superpositions arise with η^2 and *gentop*, and are easily closed up, so that it suffices to notice that β^2 cannot duplicate existing redexes (β^2 can only duplicate types, that are not redexes), so that the constraint on the R1 reduction that closes the diagram gives no problem. The details are left to the reader. \square

So we finally get, by Hindley-Rosen's Lemma.

Theorem 3.9 The systems $\xrightarrow{\beta^2 \eta^2 \pi^*}$ and $\xrightarrow{\beta^2 \eta^2}$ are confluent ⁸.

4 Weak Normalization

For the first order systems, we get from the previous section a normalizing strategy for free: first go to the *gentop* normal form, then use the SN property on *gentop* normal forms.

Summarizing, we have obtained:

Theorem 4.1 The calculi $\lambda^1 \beta \eta^*$, $\lambda^1 \beta \eta \pi^*$ are effectively weakly normalizing.

Since for the second order systems we have left out β^2 and η^2 , we find them on the way: we can deal with them at the price of a splitting of the set of rules which is different from the splitting which lead us to confluence.

Theorem 4.2 The calculi $\lambda^2 \beta \eta^*$, $\lambda^2 \beta \eta \pi^*$ are effectively weakly normalizing.

⁸ We also found an alternative proof of the confluence of $\xrightarrow{\beta^2 \eta^2}$ that does not extend to the case with SP. It relies on yet another splitting of the rules, taking *gentop* and the β rules on one hand, and the eta-like rules on the other. The proof uses the same criterion for confluence as we used in this section. In order to check the last condition, we rely on a parallelization of R2, which does not work well when the non linear surjective pairing rule is added to R2 (cf. introduction). So we abandoned that proof technique which we were not able to extend to the full system.

Proof. The reduction system R can be split into the two subsystems $R1 = \{\beta, \pi, \text{gentop}, \beta^2, \eta^2\}$ and $R2 = \{\eta, SP, \eta_{top}, SP_{top}\}$. R1 is shown to be SN by a straightforward adaptation of the technique of [GLT90] (see Section B). R2 is obviously SN since the rules strictly decrease the size of the terms they apply to. The set of R1-normal forms is closed under R2 reductions. So we get the following effective normalizing (standard) strategy.

Given a term M,

1. first R1-normalize it reaching, say, M',
2. then R2-normalize M' reaching, say, M''.

M'' is the desired normal form. \square

The previous result about weak normalization for the first order fragment can obviously be derived as a corollary from this theorem, but we actually needed the ingredients of the previous proof to get the confluence of our systems.

5 Decidability and conservative extension results

From the confluence and weak normalization for our calculi, it is now easy to get also the decidability of the associated equational theories as well as conservativity results.

Corollary 5.1 *The equational theories for $\lambda^1\beta\eta*$, $\lambda^1\beta\eta\pi*$, $\lambda^2\beta\eta*$ and $\lambda^2\beta\eta\pi*$ are decidable.*

Proof. Given terms M and N, consider their normal forms M' and N' (they exist by WN). If $M = N$, then (by CR) M' is syntactically equal to N'. So, to decide equality it suffices to take the normal forms (that is effective, as we provided a normalizing strategy for each one of these calculi) and to check if they are equal. \square

Corollary 5.2 *(Conservative extensions) For L any of the calculi $\lambda^2\beta\eta\pi*$, $\lambda^2\beta\eta*$, $\lambda^1\beta\eta\pi*$ or $\lambda^1\beta\eta*$ call \xrightarrow{L} the rewriting system corresponding to L, that is $\xrightarrow{\beta^2\eta^2\pi^*}$, $\xrightarrow{\beta^2\eta^2*}$, $\xrightarrow{\beta\eta\pi^*}$ or $\xrightarrow{\beta\eta*}$. Let L' be a subtheory of L which has the following stability property. If M is in the sublanguage of L' and $M \xrightarrow{L} N$, then N is also in L' and M and N are provably equal in L'. If M and N are terms of L' that are equal in L, then they are already equal in L'.*

Proof. If M and N are equal in L, then, by the CR property, there exist a term P s.t. M and N reduce to P in L. But M and N are terms of L', and no reduction in any of the calculi we consider can reach terms outside L', then the reductions $M \xrightarrow{L} P$ and $N \xrightarrow{L} P$ correspond to provable equations in L', so that M is equal to N in L'. \square

In [BDCL90], for example, we need the conservativity of the equational theory of $\lambda^1\beta\eta\pi^*$ over the simple typed λ -calculus, while in [DC91], we actually use the conservativity of $\lambda^2\beta\eta\pi^*$ over the second order lambda calculus.

As far as we know, our results are new for what concerns polymorphism, while other proofs of corollary 5.1 have been given in the literature, for the case of the first order calculi. We already briefly hinted at the method used in [LS86], which is based on

- the elimination of Top
- a proof of confluence via WCR and SN (WCR holds there without a need to add funny rules, and the computability method works well without special restrictions, as was first shown by R. De Vrijer).

Another method, which was found independently by A.S. Troelstra (see [Tro86], where it is used to prove SN rather than CR) and T. Hardin (see [Har89]) goes further by eliminating products as well as Top. The two methods allow to prove conservativity as well as decidability, but the overall construction is quite tedious. Let us be more specific, since the explanations provided by Lambek and Scott, in [LS86] pp. 81 and 82, are somewhat handwaving. The exploitation of the type isomorphisms can be formalized as follows. To every type T we associate a **T**-free type T^\diamond .

Definition 5.3 For any type T , we define its “top-free” form T^\diamond as the normal form of T w.r.t. the following (confluent and strongly normalizing) type rewrite system \rightsquigarrow :

$$\begin{array}{ll} A \times T \rightsquigarrow A & T \times A \rightsquigarrow A \\ T \rightarrow A \rightsquigarrow A & A \rightarrow T \rightsquigarrow T \end{array}$$

Thus a “**T**-free” type is either **T**, or a type where **T** does not occur. Then one may extend this mapping to terms, so that for a term $M:A$ we have $M^\diamond:A^\diamond$, in such a way that

$$M =_{\beta\eta\pi^*} N \iff M^\diamond =_{\beta\eta\pi} N^\diamond$$

Similarly, to a type A of $\lambda^1\beta\eta\pi^*$ we can associate a sequence of types A^* constructed from type variables with the arrow only, and to a term M a sequence M^* of terms of the types that appear in A^* . Then $M =_{\beta\eta\pi^*} N$ iff $M_1 =_{\beta\eta} N_1, \dots, M_n =_{\beta\eta} N_n$, where $M^* = M_1, \dots, M_n$ and $N^* = N_1, \dots, N_n$.

This formalizes the assertion of Lambek and Scott that there is “no loss of generality”, as far as decision is concerned, if one removes the terminal object (or both the terminal object and the products).

Moreover these translations of types and terms are conservative in the sense that if A is a type where **T** (respectively **T** and \times) does not occur, and $M:A$, then A^\diamond and M^\diamond (respectively A^* and M^*) are just A and M . Corollary 5.2 is an immediate consequence of this.

Yet another solution to the decidability problem for equational theories of cartesian closed categories has been proposed by A. Obtulowicz [Obt87]. His approach is very algebraic in nature. Obtulowicz defines effectively operations on some canonical forms, turning the set of canonical forms into an initial algebra. Then, to decide that two terms are equal, one computes their interpretation in the initial algebra, and checks whether the resulting canonical forms coincide. This approach is very technical, and contains hidden rewriting techniques. But it is interesting, because it does not a priori require such strong assumptions as to find a noetherian and confluent rewriting system.

Anyway, A. Obtulowicz did not show decidability for exactly the same equational theories as we do here. Specifically, he deals with the critical pairs which lead us to the SP_{top} rules in a different way. He forces an equational theory on types as well as on terms. Specifically, the canonical type isomorphisms underlying the translation \diamond above are forced to be true equalities (and models of these theories have thus to identify on the nose, say $A \times T$ and A). New equations between terms are added, which witness these identifications at the level of terms. Here is one of them

$$\langle M, * \rangle = M \text{ for } M : A \times T$$

With the aid of this equation and of one of its consequences, namely

$$p_1 M = M \text{ for } M : A \times T$$

one can solve the critical pair

$$\langle p_1 M, * \rangle \leftarrow \langle p_1 M, p_2 M \rangle \rightarrow M$$

by just noting that $\langle p_1 M, * \rangle \rightarrow p_1 M \rightarrow M$. It would be worthwhile to investigate these theories from a rewriting point of view.

Another treatment of the terminal object with identification of types can be found in [Nip90], which is only concerned with *local* confluence.

Let us mention that the problem of finding a confluent completion of the theory $\lambda^1\beta\eta\pi*$ has been considered in [PV87], where it was believed to be solved. Unfortunately the authors of [PV87] missed the critical pair leading to η_{top} , which in turn induced them to believe that the adaptation of the standard SN proof was straightforward.

While writing the final version of this paper, we came across [Jay91] that suggests some new ideas: by turning η and SP into expansions instead of contractions, a strongly normalizing system is obtained. It is not a rewrite system in the usual sense, though, and not even a conditional one, as in order to get termination there is the need of some restrictions on the reductions that take into account the *context* where a redex occurs.

6 Conclusion

We have established the decidability of various theories containing a rule for a terminal type, by using the classical game of Knuth-Bendix completion. We proved these results under the pressure of two quite different research works which shared the need for them.

We intend to investigate how far our method can be extended to yield similar results for other or further extensions of the λ -calculus. Actually, [PV87], for example, is concerned with a first order calculus that is $\lambda^1\beta\eta\pi^*$ plus sums: we believe that our technique applies to that case too. It would be interesting to generalize the work done here for $\lambda^2\beta\eta\pi^*$ and its subcalculi in such a way to give uniformly a CR, WN reduction system for the lambda calculi with terminal object *and* inductive types.

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Appendix: Strong normalization for subsystems

Our proof of confluence in Theorem 3.5 relies upon the strong normalization of $\xrightarrow{\beta\eta^2\pi^*}$ over the set of *gentop* normal forms, while we need the strong normalization of $\xrightarrow{\beta^2\eta^2\pi^*}$ less η_{top} and SP_{top} over the full set of terms in order to provide an effective weakly normalizing strategy for $\xrightarrow{\beta^2\eta^2\pi^*}$ in Theorem 4.2.

This appendix provides these two proofs of strong normalization in section A and B respectively, by suitably adapting one of the various versions of the reducibility method. We choose here to apply Girard’s method, following essentially the same proof plan as in [GLT90], pagg. 42-47. Since there is almost no difference in the proofs for the two systems, we will detail the first one only, and only point out the differences for the second case.

As we briefly suggested in the introduction (Section 2.3), the reducibility method fails for the full system where η_{top} and SP_{top} are allowed to freely interact with any term of the calculus: we are not able to deal in the crucial proofs of the abstraction and pairing lemmas (Lemmas A.13 and A.12) with some reductions that arise in the full system.

To rule out these reductions, one can either restrict the system to *gentop* normal forms only (this requires in turn to rule out the β^2 rule, that does not preserve *gentop* normal forms, as shown in Example 2.6), or one can simply rule out η_{top} and SP_{top} .

A Normalization without β^2 on *gentop* n.f.’s

In this section we will show that the system $\xrightarrow{\beta\eta^2\pi^*}$ (the full system $\xrightarrow{\beta^2\eta^2\pi^*}$ less β^2) is strongly normalizing over the set of *gentop* normal forms. This means that all along the proof *any gentop reduction is ruled out*, so we will not explicitly state all the time that *gentop* reductions cannot occur. Moreover, to improve readability, \longrightarrow will stand for $\xrightarrow{\beta\eta^2\pi^*}$ in this section.

Definitions

Definition A.1 (neutral terms) *A term $t:U$ is neutral iff one of the following conditions is satisfied:*

- if $U \notin Iso(T)$ and t is not an abstraction, a type abstraction or a pair,
- if $U \in Iso(T)$ (then t is $rep(U)$, as we consider only terms in *gentop* normal form).

Definition A.2 (longest reduction path for a term) *Let u be a term, then $\nu(u)$ denotes the length of the longest reduction path starting from u . Notice that, by König’s Lemma, if u is strongly normalisable, then $\nu(u)$ is finite.*

Definition A.3 A reducibility candidate of type U is a set R of terms of type U with the following properties.

CR1 if $t \in R$, then t is strongly normalisable.

CR2 if $t \in R$ and $t \rightarrow t'$, then $t' \in R$.

CR3 if t is neutral and for all t' s.t. $t \rightarrow t'$ we have that $t' \in R$, then $t \in R$.

Remark A.4 A reducibility candidate R of type U is never empty:

- if $U \in \text{Iso}(T)$, then $\text{rep}(U)$ is neutral and in normal form, hence belongs to R by (CR3).
- if $U \notin \text{Iso}(T)$, then any variable of type U is neutral and in normal form, hence belongs to R by (CR3).

Proposition A.5 The set of strongly normalizable terms of type U is a reducibility candidate.

Proof.

- (CR1) is a tautology.
- (CR2) if t is strongly normalisable, then every t' s.t. $t \rightarrow t'$ is strongly normalisable.
- (CR3) every reduction path leaving t must pass through one of the terms t' that are one step from t . Since all t' are strongly normalisable, then t is strongly normalisable also.

□

Definition A.6 (Product and Function space of reducibility candidates)

If R and S are reducibility candidates of types U and V , we can define sets $R \rightarrow S$ of terms of type $U \rightarrow V$ and $R \times S$ of terms of type $U \times V$ as follows:

- $t \in R \rightarrow S$ (of type $U \rightarrow V$) \iff
 - for all $u \in R$, $(tu) \in S$ if $V \notin \text{Iso}(T)$
 - $t = \text{rep}(U \rightarrow V)$ if $V \in \text{Iso}(T)$
- $t \in R \times S$ (of type $U \times V$) \iff
 - $p_1t \in R$ and $p_2t \in S$ if U, V are not in $\text{Iso}(T)$
 - $p_1t \in R$ if $U \notin \text{Iso}(T)$, $V \in \text{Iso}(T)$

- $p_2t \in S$ if $U \in Iso(T)$, $V \notin Iso(T)$
- $t = rep(U \times V)$ if $U, V \in Iso(T)$

Remark A.7 Notice that, as t and u are in *gentop* normal form, and due to the conditions on U and V , the terms (tu) , p_1t and p_2t above are still in *gentop* normal form.

Theorem A.8 If R_1 and R_2 are reducibility candidates of types U_1 and U_2 , then $R_1 \times R_2$ and $R_1 \rightarrow R_2$ are reducibility candidates of type $U_1 \times U_2$ and $U_1 \rightarrow U_2$ respectively.

Proof. Assume that R_1 and R_2 are reducibility candidates of type U_1 and U_2 , respectively.

1. $R_1 \times R_2$ is a reducibility candidate of type $U_1 \times U_2$.
 If $U_1 \times U_2 \in Iso(T)$, then (CR1), (CR2) and (CR3) hold vacuously due to the fact that we consider only *gentop* normal forms, so let's assume in the following that $U_1 \notin Iso(T)$ and/or $U_2 \notin Iso(T)$.
 - (CR1) if $t \in U_1 \times U_2$ and $U_i \notin Iso(T)$, then $p_i t$ is strongly normalisable by the induction hypothesis on U_i , since $p_i t \in U_i$ by definition. Hence t is strongly normalisable.
 - (CR2) if $t \rightarrow t'$, then $p_1 t \rightarrow p_1 t'$ and/or $p_2 t \rightarrow p_2 t'$. As $t \in U_1 \times U_2$, then $p_1 t \in U_1$ and/or $p_2 t \in U_2$. By induction hypothesis **CR2** for U_1 and/or U_2 we get $p_1 t' \in U_1$ and/or $p_2 t' \in U_2$, hence, by definition, $t' \in U_1 \times U_2$.
 - (CR3) t is neutral and all t' one step from t are in $U_1 \times U_2$.
 We need to show $p_1 t \in U_1$ and/or $p_2 t \in U_2$.
 Now notice that applying a conversion inside $p_i t$ can only result in some $p_i t'$ as t is not a pair (it is neutral and it is not $rep(U_1 \times U_2)$). But $p_1 t' \in U_1$ and/or $p_2 t' \in U_2$ as t' is in $U_1 \times U_2$. In any case, $p_1 t$ and/or $p_2 t$ are neutral and every term one step from it is in $U_1 \times U_2$, so the induction hypothesis for U_1 and/or U_2 ensure $p_1 t \in U_1$ and/or $p_2 t \in U_2$. So $t \in U_1 \times U_2$.
2. $R_1 \rightarrow R_2$ is a reducibility candidate of type $U_1 \rightarrow U_2$.

We can assume that $U_2 \notin Iso(T)$ as otherwise $U_1 \rightarrow U_2 \in Iso(T)$, and then (CR1), (CR2) and (CR3) hold vacuously.

- (CR1) if $t \in U_1 \rightarrow U_2$, then let u be a variable x of type U_1 if $U_1 \notin Iso(T)$ or else $rep(U_1)$. Since $u \in any$ reducibility candidate,

(remark A.4), we get that $(tu) \in U_2$ by definition, hence (tu) is strongly normalisable by induction hypothesis for U_2 , that suffices to show that t is strongly normalisable.

- (CR2) if $t \longrightarrow t'$, we need to show $(t'u) \in U_2$ for all $u \in U_1$. Take then $u \in U_1$; we have $(tu) \in U_2$ and $(tu) \longrightarrow (t'u)$, hence $(t'u) \in U_2$ by induction hypothesis on U_2 .
- (CR3) t is neutral and all t' one step from t are in $R_1 \rightarrow R_2$. In order to show $t \in U_1 \rightarrow U_2$, we need to show $(tu) \in U_2$ for all $u \in U_1$. By induction hypothesis on U_1 , we get u is strongly normalisable, so we can argue by induction on $\nu(u)$.

In one step, (tu) converts to:

- $(t'u)$ with t' one step from t .
As $t' \in U_1 \rightarrow U_2$, we get $(t'u) \in U_2$ by definition.
- (tu') with u' one step from u .
By induction hypothesis on U_1 , $u' \in U_1$ and $\nu(u') < \nu(u)$, so $(tu') \in U_2$ by the induction hypothesis on u .
- there is no other possibility, as t is already in *gentop* n.f. and it is neutral, hence not of the form $\lambda x.v$ (it cannot be $rep(U_1 \rightarrow U_2)$ as we already assumed $U_1 \rightarrow U_2 \notin Iso(T)$).

□

A.1 Reducibility with parameters

Let T be a type, and \vec{X} be a set of type variables containing at least all the free type variables of T . For \vec{U} a sequence of types of the same length, let $T[\vec{U}/\vec{X}]$ be the type obtained by simultaneous substitution of the X 's with the U 's, and \vec{R} a sequence of reducibility candidates of corresponding types.

Definition A.9 *The set $RED_T[\vec{R}/\vec{X}]$ of reducible terms of type $T[\vec{U}/\vec{X}]$ is defined by induction on the type T as follows.*

- if T is atomic, $RED_T[\vec{R}/\vec{X}]$ is the set of strongly normalizable terms of type $T[\vec{U}/\vec{X}] = T$
- if T is X_i , $RED_T[\vec{R}/\vec{X}]$ is R_i
- if T is $U \times V$ then $RED_T[\vec{R}/\vec{X}]$ is $RED_U[\vec{R}/\vec{X}] \times RED_V[\vec{R}/\vec{X}]$
- if T is $U \rightarrow V$ then $RED_T[\vec{R}/\vec{X}]$ is $RED_U[\vec{R}/\vec{X}] \rightarrow RED_V[\vec{R}/\vec{X}]$
- if T is $\forall Y.W$ then $RED_T[\vec{R}/\vec{X}]$ is the set of terms t of type $[\vec{U}/\vec{X}]$ such that, for every type V and reducibility candidate S of this type, $t[V] \in RED_W[\vec{R}/\vec{U}, S/Y]$

Lemma A.10 $rep(U)$ is normal for all $U \in Iso(T)$.

Proof. By a straightforward induction on the structure of the term. \square

Theorem A.11 $RED_T[\vec{R}/\vec{X}]$ is a reducibility candidate of type $T[\vec{U}/\vec{X}]$

Proof. We proceed by structural induction on the type T .

Since we consider only terms in *gentop* normal form, there is no term of type U besides $rep(U)$ if $U \in Iso(T)$. Moreover, due to the previous lemma and the definition of reducibility, $rep(U)$ trivially satisfies (CR1), (CR2) and (CR3), so we will not consider explicitly the case of types in $Iso(T)$ in the induction.

Atomic types

If T is atomic, then $RED_T[\vec{R}/\vec{X}]$ is the set of strongly normalizing terms of type T , and we already proved it to be a reducibility candidate (Proposition A.5).

Type Variables

If T is X_i , then $RED_T[\vec{R}/\vec{X}]$ is R_i , that is a reducibility candidate by definition.

Product types

Let T be $U_1 \times U_2$. Then $RED_{U_1 \times U_2}[\vec{R}/\vec{X}] = RED_{U_1}[\vec{R}/\vec{X}] \times RED_{U_2}[\vec{R}/\vec{X}]$ by definition. We can apply the induction hypothesis for $RED_{U_1}[\vec{R}/\vec{X}]$ and $RED_{U_2}[\vec{R}/\vec{X}]$, so that the result then follows by Theorem A.8.

Arrow types

Let T be $U_1 \rightarrow U_2$. Then $RED_{U_1 \rightarrow U_2}[\vec{R}/\vec{X}] = RED_{U_1}[\vec{R}/\vec{X}] \rightarrow RED_{U_2}[\vec{R}/\vec{X}]$ by definition. We can apply the induction hypothesis for $RED_{U_1}[\vec{R}/\vec{X}]$ and $RED_{U_2}[\vec{R}/\vec{X}]$, so that the result then follows by Theorem A.8.

Universal types

Let $T = \forall Y.W$. We can assume that $W \notin Iso(T)$ as otherwise $\forall Y.W \in Iso(T)$.

- (CR1) if $t \in RED_{\forall Y.W}[\vec{R}/\vec{X}]$, then let V be an arbitrary type and S be an arbitrary reducibility candidate of this type (for example, the strongly normalizable terms of type V). Then $t[V] \in RED_W[\vec{R}/\vec{X}, S/Y]$, and so, by induction hypothesis, we know that $t[V]$ is strongly normalizable. A fortiori t is strongly normalisable.
- (CR2) if $t \xrightarrow{\beta\eta\pi^*} t'$, then for all types V and reducibility candidate S of this type, we have that $t[V] \in RED_W[\vec{R}/\vec{X}, S/Y]$ and $(t[V]) \xrightarrow{\beta\eta\pi^*} (t'[V])$,

hence $t'[V] \in RED_W[\vec{R}/\vec{X}, S/Y]$ by induction hypothesis on W . So, by definition, $t' \in RED_{\forall Y.W}[\vec{R}/\vec{X}]$

- (CR3) t is neutral and all t' one step from t are in $RED_T[\vec{R}/\vec{X}]$. Take V and S : if we apply a conversion inside $t[V]$, the result is $t'[V]$ since t is neutral (and, again, not $rep(\forall Y.W)$, as $t \xrightarrow{\beta\eta\pi^*} t'$). Now, $t'[V]$ is in $RED_W[\vec{R}/\vec{X}, S/Y]$ as t' is in $RED_T[\vec{R}/\vec{X}]$. By induction hypothesis, we get $t'[V] \in RED_W[\vec{R}/\vec{X}, S/Y]$, so $t \in RED_T[\vec{R}/\vec{X}]$.

□

Reducibility theorem

We shall need some lemmas to deduce reducibility of a term from reducibility of its subterms.

Lemma A.12 (Pairing)

If $u_1 \in RED_{U_1}[\vec{R}/\vec{X}]$ and $u_2 \in RED_{U_2}[\vec{R}/\vec{X}]$, then $\langle u_1, u_2 \rangle \in RED_{U_1 \times U_2}[\vec{R}/\vec{X}]$.

Proof. We can assume that $U_1 \notin Iso(T)$ and/or $U_2 \notin Iso(T)$, as otherwise $\langle u_1, u_2 \rangle = rep(U_1 \times U_2)$ and then $RED_{U_1 \times U_2}[\vec{R}/\vec{X}]$ is $\{rep(U_1 \times U_2)\}$.

We can argue by induction on $\nu(u_1) + \nu(u_2)$, by CR1, to show that, for $i=1$ and/or $i=2$, $p_i \langle u_1, u_2 \rangle \in RED_{U_i}[\vec{R}/\vec{X}]$.

Let $i=1$ for simplicity. The term $p_1 \langle u_1, u_2 \rangle$ converts to:

- u_1 , which is in $RED_{U_1}[\vec{R}/\vec{X}]$ by hypothesis.
- $p_1 \langle u', u_2 \rangle$ with u' one step from u_1 .
Then u' is in $RED_{U_1}[\vec{R}/\vec{X}]$ by CR2 and $\nu(u') < \nu(u_1)$, so $p_1 \langle u', u_2 \rangle \in RED_{U_1}[\vec{R}/\vec{X}]$ by induction hypothesis.
- $p_1 \langle u_1, v' \rangle$ with v' one step from u_2 . We get $p_1 \langle u_1, v' \rangle \in RED_{U_1}[\vec{R}/\vec{X}]$ as above.
- $p_1 w$ if u_1 is $p_1 w$ and u_2 is $p_2 w$.
But $p_1 w = u_1$ is in $RED_{U_1}[\vec{R}/\vec{X}]$ by hypothesis.
- $p_1 w$ if u_1 is $p_1 w$ and u_2 is $rep(U_2)$.
By definition of parametric reducibility for product types when one of the factor types is in $Iso(T)$, we get that $u_1 \in RED_{U_1}[\vec{R}/\vec{X}]$ as $p_1 w = u_1$ is in $RED_{U_1}[\vec{R}/\vec{X}]$ by hypothesis.

In every case, the neutral terms $p_i\langle u_1, u_2 \rangle$ convert to terms in $RED_{U_i}[\vec{R}/\vec{X}]$ only, for $i=1$ and/or $i=2$, so they are in $RED_{U_i}[\vec{R}/\vec{X}]$ by CR3. Hence $\langle u_1, u_2 \rangle$ is in $RED_{U_1 \times U_2}[\vec{R}/\vec{X}]$. \square

Lemma A.13 (*Abstraction*)

Let $x:U$ and $v:V$. If for all $u \in RED_U[\vec{R}/\vec{X}]$ $v[u/x] \in RED_V[\vec{R}/\vec{X}]$, then $\lambda x.v \in RED_{U \rightarrow V}[\vec{R}/\vec{X}]$

Proof. We can assume that $V \notin Iso(T)$ as otherwise v is $rep(V)$, and $\lambda x.v$ is $rep(U \rightarrow V)$ as $U \rightarrow V \in Iso(T)$, and it is reducible by definition.

To show that $\lambda x.v \in RED_{U \rightarrow V}[\vec{R}/\vec{X}]$, we need to show that $(\lambda x.v)u \in RED_V[\vec{R}/\vec{X}]$ for all $u \in RED_U[\vec{R}/\vec{X}]$.

There are two cases: either $U \in Iso(T)$ or not.

In the first case, $v[u/x] = v$ as it is in *gentop* normal form, hence there is no free occurrence of x in v , and the only term u of type U is $rep(U)$. Since $t = (\lambda x.v)u$ is neutral, it suffices to show that for every term t' one step from it $t' \in RED_V[\vec{R}/\vec{X}]$. Since $v = v[rep(U)/x] \in RED_V[\vec{R}/\vec{X}]$ by hypothesis, hence strongly normalizing, we can argue by induction on $\nu(v)$. The one step reducts of $(\lambda x.v)u$ are:

- $v[u/x]$ which is in $RED_V[\vec{R}/\vec{X}]$ by hypothesis
- $(\lambda x.v')u$ with v' one step from v . Then $v'[u/x]$ is in $RED_V[\vec{R}/\vec{X}]$ by CR2 as it is one step from $v[u/x]$ and we are done by induction hypothesis as $\nu(v') < \nu(v)$
- $(v'u)$ via η_{top} if $v = v'rep(U)$.
Now, $u = rep(U)$ so $(v'u) = v'rep(U) = v = v[u/x]$ which is in $RED_V[\vec{R}/\vec{X}]$ by hypothesis.

In the second case, $x:U$ is in $RED_U[\vec{R}/\vec{X}]$ (remark A.4). So $v = v[x/x]$ is in $RED_V[\vec{R}/\vec{X}]$, hence strongly normalizable by CR2 and we can argue by induction on $\nu(u) + \nu(v)$ to show that all terms one step from $(\lambda x.v)u$ are reducible.

The one step reducts of $(\lambda x.v)u$ are:

- $v[u/x]$ that is in $RED_V[\vec{R}/\vec{X}]$ by hypothesis.
- $(\lambda x.v')u$ with v' one step from v . Since $v'[u/x]$ is one step from $v[u/x]$ ⁹, then it is in $RED_V[\vec{R}/\vec{X}]$ by CR2. Furthermore, $\nu(v') < \nu(v)$, so by

⁹Can be shown by an easy induction on v .

induction hypothesis we get $(\lambda x.v' u) \in RED_V[\vec{R}/\vec{X}]$.

- $(\lambda x.v)u'$ with u' one step from u . Then $u' \in RED_U[\vec{R}/\vec{X}]$ by CR2, $\nu(u') < \nu(u)$ and $v[u'/x] \in RED_V[\vec{R}/\vec{X}]$ by repeated applications of CR2, as it is some step from $v[u/x]$. So we can apply again the induction hypothesis.
- $(v'u)$ via η if $\lambda x.v$ is $\lambda x.v'x$ and $x \notin FV(v')$.
It is in $RED_V[\vec{R}/\vec{X}]$ as $v[u/x] = (v'u)$ is in $RED_V[\vec{R}/\vec{X}]$ by hypothesis.

Since $(\lambda x.v)u$ is neutral and it converts to reducible terms only, it is reducible. Hence $\lambda x.v$ is reducible. \square

Remark A.14 *Working only with terms in gentop normal form allows us to rule out all the other reductions that are possible when considering all the terms of the calculus. This restriction is essential as otherwise we ought now to face, in Lemma ??, reductions like $p_1(\text{rep}(U_1), p_2w) \rightarrow p_1w$, that we cannot handle, as nothing in our induction hypothesis allows us to conclude that p_1w is reducible. (We already pointed out the difficulty in Section 2.3). This reduction¹⁰ is now ruled out as $p_1(\text{rep}(U_1), p_2w)$ is not a gentop normal form (its normal form being $\text{rep}(U_1)$). Similarly, in Lemma A.13, the restriction to terms in gentop normal form allows us to rule out (in the case $U \in \text{Iso}(T)$) all the other reductions otherwise possible in the full calculus. As pointed out in the introduction (Section 2.3), we do not know how to handle the general reduction $(\lambda x.(v' \text{rep}(U)))u \rightarrow (v' u)$ via η_{top} : if u is not $\text{rep}(U)$, then we have nothing in our induction hypothesis to tell us that $(v' u)$ is reducible. But here u must be in gentop normal form, that is to say, $u = \text{rep}(U)$, and the η_{top} reduction can be handled as above.*

Lemma A.15 *(Universal abstraction)*

If for every type V and candidate S of type V , $v[V/Y] \in RED_W[\vec{R}/\vec{X}, S/Y]$, then $\Lambda Y.v \in RED_{\forall Y.W}[\vec{R}/\vec{X}]$

Proof. We need to show that $(\Lambda Y.v)[V] \in RED_W[\vec{R}/\vec{X}, S/Y]$ for every type V and candidate S of type V . We argue by induction on $\nu(v)$, using the fact that $(\Lambda Y.v)[V]$ is neutral. Converting a redex of $(\Lambda Y.v)[V]$ can yield:

- $(\Lambda Y.v')[V]$ with v' one step from v ; now, $(\Lambda Y.v')[V] \in RED_W[\vec{R}/\vec{X}, S/Y]$ by induction hypothesis on $\nu(v)$

The results follows by CR3. \square

Lemma A.16 $RED_{T[V/Y]}[\vec{R}/\vec{X}] = RED_T[\vec{R}/\vec{X}, RED_V[\vec{R}/\vec{X}]/Y]$

¹⁰ And its symmetric $p_2(p_1w, \text{rep}(U_2)) \rightarrow p_2w$.

Proof. By induction on T . \square

Lemma A.17 (*Universal application*)

If $t \in RED_{\forall Y.W}[\vec{R}/\vec{X}]$, then $t[V] \in RED_{W[V/Y]}[\vec{R}/\vec{X}]$ for every type V .

Proof. By hypothesis, $t[V] \in RED_W[\vec{R}/\vec{X}, S/Y]$ for every candidate S . Taking $S = RED_V[\vec{R}/\vec{X}]$, the result follows by Lemma A.16. \square

The theorem

As in [GLT90], we say here that a term t of type T is *reducible* if it is in $RED_T[\vec{S}/\vec{X}]$, where \vec{X} are the free type variables of T and SN_i is the set of strongly normalizable terms of type X_i .

In the proof of the theorem, there is the need of a stronger induction hypothesis, from which the strong normalization follows by putting $u_i = x_i$ and $R_i = SN_i$.

Proposition A.18 *Let $t:T$ be any term (in gentop normal form) of $\lambda^2\beta\eta\pi*$, whose free variables are among $x_1 : U_1 \dots x_n : U_n$, and all the free variable of T , U_1, \dots, U_n are among X_1, \dots, X_m . If R_1, \dots, R_m are reducibility candidates of types V_1, \dots, V_m , and u_1, \dots, u_m are terms of types $U_1[\vec{V}/\vec{X}], \dots, U_m[\vec{V}/\vec{X}]$ which are in $RED_{U_1}[\vec{R}/\vec{X}], \dots, RED_{U_m}[\vec{R}/\vec{X}]$, then $t[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_T[\vec{R}/\vec{X}]$.*

Proof. By induction on t . Notice that there are no variables of type U if $U \in Iso(T)$.

- $t = *$: t is in the only reducibility candidate $\{*\}$ of type \mathbf{T} .
- $t = x_i$: in this case the statement of the theorem becomes a tautology.
- $t = p_i u$: then $u:U_1 \times U_2$ and $U_i \notin Iso(T)$ as we consider only terms in *gentop* normal form. By induction hypothesis, $u[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{U_1 \times U_2}[\vec{R}/\vec{X}]$. Hence $(p_i u)[\vec{V}/\vec{X}][\vec{u}/\vec{x}] = p_i u[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{U_i}[\vec{R}/\vec{X}]$ by definition of reducibility for product types.
- $t = \langle u, v \rangle$: since $u[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{U_1}[\vec{R}/\vec{X}]$ and $v[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{U_2}[\vec{R}/\vec{X}]$ by the induction hypothesis, Lemma A.12 gives $\langle u[\vec{V}/\vec{X}][\vec{u}/\vec{x}], v[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \rangle \in RED_{U_1 \times U_2}[\vec{R}/\vec{X}]$. But $\langle u, v \rangle[\vec{V}/\vec{X}][\vec{u}/\vec{x}]$ is $\langle u[\vec{V}/\vec{X}][\vec{u}/\vec{x}], v[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \rangle$ by definition, hence $\langle u, v \rangle[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{U_1 \times U_2}[\vec{R}/\vec{X}]$.

- $t = \lambda z.v$: by induction hypothesis, we know that $v[\vec{V}/\vec{X}][\vec{u}/\vec{x}][u/z] \in RED_V[\vec{R}/\vec{X}]$ for all $u \in RED_U[\vec{R}/\vec{X}]$. Then Lemma A.13 gives $\lambda z.v[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{U \rightarrow V}[\vec{R}/\vec{X}]$.
But $(\lambda z.v)[\vec{V}/\vec{X}][\vec{u}/\vec{x}]$ is $\lambda z.v[\vec{V}/\vec{X}][\vec{u}/\vec{x}]$ by definition, and the result follows.
- $t = vu$: then $v[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{U \rightarrow V}[\vec{R}/\vec{X}]$ and $u[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_U[\vec{R}/\vec{X}]$ by induction hypothesis.
Hence $(v[\vec{V}/\vec{X}][\vec{u}/\vec{x}] u[\vec{V}/\vec{X}][\vec{u}/\vec{x}]) \in RED_V[\vec{R}/\vec{X}]$, as it is $(vu)[\vec{V}/\vec{X}][\vec{u}/\vec{x}]$ by definition.
- $t = \Lambda Y.v$: then we know by induction hypothesis that for every type V and reducibility candidate S we have $v[V/Y][\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_W[\vec{R}/\vec{X}, S/Y]$. Then Lemma A.15 yields the result $(\Lambda Y.v)[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{\forall Y.W}[\vec{R}/\vec{X}]$
- $t = t[V]$: then we know by induction hypothesis that $t[\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{\forall Y.W}[\vec{R}/\vec{X}]$ and Lemma A.17 gives the result $t[V][\vec{V}/\vec{X}][\vec{u}/\vec{x}] \in RED_{W[V/Y]}[\vec{R}/\vec{X}]$ for every type V .

□

Theorem A.19 $\xrightarrow{\beta\eta^2\pi^*}$ is strongly normalizing over the set of *gentop* normal forms.

Proof. Let t be any term in *gentop* normal form. All its free variables are in any reducibility candidate by CR3, so that $t = t[\vec{S}\vec{N}/\vec{X}][\vec{x}/\vec{x}]$ is reducible by the previous lemma. By CR1 it is strongly normalizing. That is, $\xrightarrow{\beta\eta^2\pi^*}$ is strongly normalizing over *gentop* normal forms. □

B Normalization without η_{top} and SP_{top}

The proof of strong normalization is essentially the same as the one given above for the full system without β^2 over the subset of terms in *gentop* normal form.

The main difference, besides the fact that we add β^2 and *gentop* and exclude η_{top} and SP_{top} , is that now we work on the full set of terms, so that there are plenty of terms $t:U$, besides $rep(U)$, when $U \in Iso(T)$: we keep essentially the same notion of neutral term (A.1), but it is to be noted that only $rep(U)$ is neutral, not every term of type $U \in Iso(T)$.

Definition B.1 (neutral terms) A term $t:U$ is neutral iff at least one of the following conditions is satisfied:

- if $U \notin Iso(T)$ and t is not an abstraction, a type abstraction or a pair,
- if $U \in Iso(T)$ and t is $rep(U)$.

Since we drop η_{top} and SP_{top} , there is no need to give a special status to the types $U \in Iso(T)$ (besides the fact that $rep(U)$ is neutral), and we resort to the usual definition of product and function space of reducibility candidates, that allows us to deal with all the terms of type $U \in Iso(T)$.

Definition B.2 (Product and Function space of reducibility candidates)

If R and S are reducibility candidates of types U and V , we define:

- $t \in R \rightarrow S \iff$ for all $u \in R$, $tu \in S$
- $t \in R \times S \iff p_1t \in U$ and $p_2t \in V$

With this new definition, the proofs of the previous appendix go through almost unchanged, with the only care to keep in mind that now $rep(U)$ is no longer the only term of type $U \in Iso(T)$, and that types in $Iso(T)$ have no longer a special status. This means that wherever there is a distinction between types that are in $Iso(T)$ and types that are not, one follows the proof given for types that are not in $Iso(T)$. The new cases arising from *gentop* reductions are easily dealt with, as $rep(U)$ is still in any reducibility candidate by CR3.

For completeness, we detail here all the changes that are needed.

- Remark A.4 now extends to *all* variables, also the variables of type $U \in Iso(T)$. It is just the matter of noticing that a variable $x:U \in Iso(T)$ is neutral and reduces only to $rep(U)$, that is in any reducibility candidate by CR3, and the result follows by CR3.
- in Theorem A.8, we can no longer factor out the types in $Iso(T)$, that must be treated exactly as the other types:

Product Types (CR3)

- * t can be $rep(U_1 \times U_2)$. In that case the only possible reduction for $p_i t$ (that is not in *gentop* normal form) is to $rep(U_i)$, that is in any reducibility candidate (remark A.4), hence in $RED_{U_i}[\vec{R}/\vec{X}]$ that is a reducibility candidate by induction hypothesis on U_i . So $p_i t \in RED_{U_i}[\vec{R}/\vec{X}]$ by CR3 on U_i and we get $t \in RED_{U_1 \times U_2}[\vec{R}/\vec{X}]$ by definition.
- * t can be a neutral term different from $rep(U_1 \times U_2)$. Then the only possible reduction for $p_i t$ (that is not in *gentop* normal form) is to $rep(U_i)$, and we conclude as above.

Arrow Types (CR3)

- * t (or t') can be $rep(U_1 \rightarrow U_2)$. Then (tu) (or $(t'u)$) can only reduce to $rep(U_2)$ that is in any reducibility candidate (remark A.4), hence in $RED_{U_2}[\vec{R}/\vec{X}]$ that is a reducibility candidate by induction hypothesis on U_2 . So (tu) (or $(t'u)$) $\in RED_{U_2}[\vec{R}/\vec{X}]$ for all $u \in RED_{U_1}[\vec{R}/\vec{X}]$ and we get $t \in RED_{U_1 \rightarrow U_2}[\vec{R}/\vec{X}]$ by definition.
 - * t can be a neutral term different from $rep(U_1 \rightarrow U_2)$. Then the only possible reduction for (tu) (or $(t'u)$) is to $rep(U_2)$, and we conclude as above.
- in Theorem A.11, we can no longer factor out the types in $Iso(T)$, that must be treated exactly as the other types.

Universal Types (CR3)

- * t (or t') can be $rep(\forall Y.W)$. Then $t[V]$ can only reduce to $rep(W)$, that is in any reducibility candidate (Remark A.4), hence in $RED_W[\vec{R}/\vec{X}]$ that is a reducibility candidate by induction hypothesis on W . Again we get t (or t') $\in RED_{\forall Y.W}[\vec{R}/\vec{X}]$ by definition.
 - * t (or t') can be a neutral term different from $rep(\forall Y.W)$. Then $t[V]$ can only reduce to $rep(W)$, and we conclude as above.
- in Lemma A.12 and A.13 we can no longer factor out the case of types $U \in Iso(T)$, that must be treated uniformly as the other types. Since the rules SP_{top} and η_{top} are not present, only the first four cases considered in Lemma A.12 can occur and the proof goes through unchanged for them, while for Lemma A.13 we follow the proof given for $V \notin Iso(T)$.

There is now the further possibility of a *gentop* reduction, that is in both cases dealt with in the usual way by remembering that *any* reducibility candidate of type $U \in Iso(T)$ contains $rep(U)$.

- in Lemma A.15 we have now two additional cases:
 - $(\Lambda Y.v)[V]$ reduces to $rep(W[V/Y])$, that is in $RED_{W[V/Y]}[\vec{R}/\vec{X}]$ since this latter is a reducibility candidate.
 - $(\Lambda Y.v)[V]$ reduces to $v[V/Y]$. But we know that $v[V/Y] \in RED_{W[V/Y]}[\vec{R}/\vec{X}, S/Y]$ by hypothesis.
- In the proof of the Proposition A.18, it suffices to apply to the types $V \in Iso(T)$ the same arguments used for types $U \notin Iso(T)$, as now there is no longer any difference in the definition of the function space and product of reducibility candidates.

Using again the fact that $t = t[\vec{S}\vec{N}/\vec{X}][\vec{x}/\vec{x}]$, we similarly get our final result.

Theorem B.3 $\xrightarrow{\beta^2 \eta^2 \pi^*}$ *without η_{top} and SP_{top} is strongly normalizing.*

M	LHS	RHS	Comment
x	$(N)^T$	$(N)^T$	ok
y	y	y	ok
(PQ)	$((PQ)[N/x])^T$ $(P[N/x]Q[N/x])^T$ $= ((P[N/x])^T(Q[N/x])^T)$ $= ((P)^T[(N)^T/x])(Q)^T[(N)^T/x]$	$(PQ)^T[(N)^T/x]$ $= ((P)^T(Q)^T)[(N)^T/x]$ $= ((P)^T[(N)^T/x](Q)^T[(N)^T/x])$	by def. of subst. by def. of subst. by def. of subst. by ind. hyp.
$\lambda y.P$	$(\lambda y.P[N/x])^T$ $\lambda y.(P[N/x])^T$ $= \lambda y.(P)^T[(N)^T/x]$	$(\lambda y.P)^T[(N)^T/x]$ $(\lambda y.(P)^T)[(N)^T/x]$ $= \lambda y.(P)^T[(N)^T/x]$	by def. of subst. by def. of subst. by ind. hyp.
$p_i P$	$(p_i P[N/x])^T$ $= p_i(P[N/x])^T$ $= p_i(P)^T[(N)^T/x]$	$(p_i P)^T[(N)^T/x]$ $= p_i(P)^T[(N)^T/x]$ $= p_i(P)^T[(N)^T/x]$	by def. of subst. by def. of subst. by ind. hyp.
$\langle P, Q \rangle$	$(\langle P[N/x], Q[N/x] \rangle)^T$ $= \langle (P[N/x])^T, (Q[N/x])^T \rangle$ $= \langle (P)^T[(N)^T/x], (Q)^T[(N)^T/x] \rangle$	$(\langle P, Q \rangle)^T[(N)^T/x]$ $= \langle (P)^T, (Q)^T \rangle[(N)^T/x]$ $= \langle (P)^T[(N)^T/x], (Q)^T[(N)^T/x] \rangle$	by def. of subst. by def. of subst. by ind. hyp.
$\Lambda t.P$	$(\Lambda t.P[N/x])^T$ $\Lambda t.(P[N/x])^T$ $= \Lambda t.(P)^T[(N)^T/x]$	$(\Lambda t.P)^T[(N)^T/x]$ $(\Lambda t.(P)^T)[(N)^T/x]$ $= \Lambda t.(P)^T[(N)^T/x]$	by def. of subst. by def. of subst. by ind. hyp.
$P[A]$	$(P[A][N/x])^T$ $= (P[N/x][A])^T$ $= (P[N/x])^T[A]$ $= (P)^T[(N)^T/x][A]$	$(P[A])^T[(N)^T/x]$ $= (P)^T[A][(N)^T/x]$ $= (P)^T[(N)^T/x][A]$	by def. of subst. by def. of subst. by def. of subst. by ind. hyp.

Table 1: Compatibility of *gentop* n.f. with substitution.