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Proof Nets and Explicit Substitutions

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We refine the simulation technique introduced in (Di Cosmo and Kesner 1997) to show strong normalization of λ-calculi with explicit substitutions via termination of cut elimination in proof nets (Girard 1987). We first propose a notion of equivalence relation for proof nets that extends the one in (Di Cosmo and Guerrini 1999), and we show that cut elimination modulo this equivalence relation is terminating. We then show strong normalization of the typed version of the λ_{ws}-calculus with de Bruijn indices (a calculus with full composition defined in (David and Guillaume 1999)) using a translation from typed λ_{ws} to proof nets. Finally, we propose a version of typed λ_{ws} with named variables which helps to better understand the complex mechanism of the explicit weakening notation introduced in the λ_{ws}-calculus with de Bruijn indices (David and Guillaume 1999).

1. Introduction

This paper uses linear logic's proof nets, equipped with an extended notion of reduction, to provide several new results in the field of explicit substitutions. It is also an important step forward in clarifying the connection between explicit substitutions and proof nets, two well established formalisms that have been used to gain a better understanding of the λ-calculus over the past decade. On one side, explicit substitutions provide an intermediate formalism that - by decomposing the β rule into more atomic steps - allows a better understanding of the execution models. On the other side, linear logic decomposes the intuitionistic logical connectives, like the arrow, into more atomic, resource-aware connectives, like the linear arrow and the explicit erasure and duplication operators given by the exponentials: this decomposition is reflected in proof nets, which are the computational side of linear logic, and provides a more refined computational model than the one given by the λ-calculus, which is the computational side of intuitionistic logic[†].

The pioneer calculus with explicit substitutions, λ_σ, was introduced in (Abadi *et al* 1991,) as a bridge between the classical λ-calculus and concrete implementations of functional programming languages. An important property of calculi with explicit substitutions is

[†] Using various translations of the λ-calculus into proof nets, new abstract machines have been proposed, exploiting the Geometry of Interaction and the Dynamic Algebras (Girard 1989; Abramsky and Jagadeesan 1992; Danos 1990), leading to the works on optimal reduction (Gonthier, Abadi Lévy 1992; Lamping 1990).

nowadays known as PSN, which stands for "Preservation of Strong Normalization": a calculus with explicit substitutions has PSN when all λ-terms that are strongly normalizing using the traditional β-reduction rule are also strongly normalizing w.r.t. the more refined reduction system defined using explicit substitutions. But λ_σ does *not* preserve β-strong normalization as shown by Mellies, who exhibited a well-typed term which, due to the substitution composition rules in λ_σ, is not λ_σ-strongly normalizing (Melliès 1995).

Since then, a quest was started to find an "optimal" calculus having all of a wide range of desired properties: it should preserve strong normalization, but also be confluent (in a very large sense that implies the ability to compose substitutions), and its typed version should be strongly normalizing.

Meanwhile, in the linear logic community, many studies focused of the connection between λ-calculus (without explicit substitutions) and proof nets, trying to find the proper variant or extension of proof nets that could be used to cleanly simulate β-reduction, like in (Danos and Regnier 1995).

Finally, in (Di Cosmo and Kesner 1997), the first two authors of this work showed for the first time that explicit substitutions could be tightly related to linear logic's proof nets, by providing a translation into a variant of proof nets from $\lambda\mathtt{x}$ (Rose 1992; Bloo and Rose 1995), a simple calculus with explicit substitutions and named variables, but no composition.

This connection was promising because proof nets seem to have many of the properties which are required of a "good" calculus of explicit substitutions, and especially the strong normalization in the presence of a reduction rule which is reminiscent of the composition rule at the heart of Mellies' counterexample. But (Di Cosmo and Kesner 1997) only dealt with a calculus without composition, and the translation was complex and obscure enough to make the task of extending it to the case of a calculus with composition quite a daunting one.

In this paper, we can finally present a notion of reduction for Girard's proof nets which is flexible enough to allow a natural and simple translation from David and Guillaume's λ_{ws}, a complex calculus of explicit substitution with de Bruijn indices and full composition (David and Guillaume 1999; David and Guillaume 2001). This translation allows us to prove that typed λ_{ws} is strongly normalizing, which is a new result confirming a conjecture in (David and Guillaume 1999; David and Guillaume 2001). Also, the fact that in the translation all information about variable order is lost suggests a version of typed λ_{ws} with named variables which is immediately proved to be strongly normalizing. This is due to the fact that only the type information is used in the translation of both calculi. Also, we believe that the typed named version of λ_{ws} gives a better understanding of the mechanisms of labels existing in the calculus. In particular, names allow to understand the fine manipulation of explicit weakenings in λ_{ws} without entering into the complicate details of renaming used in a de Bruijn setting.

The paper is organized as follows: we first recall the basic definitions of linear logic and proof nets and we introduce our refined reduction system for proof nets (Section 2), then prove that it is strongly normalizing (Section 3). In Section 4 we recall the definition of the λ_{ws} calculus with its type system, present the translation into proof nets, and show strong normalization of typed λ_{ws}. Finally, we introduce a version of typed λ_{ws} with named variables (Section 5), enjoying the same good properties, and we conclude with some remarks and directions for future work (Section 7).

2. Linear logic, proof nets and extended reduction

We recall here some classical notions from linear logic, namely the linear sequent calculus and proof nets, and some basic results concerning confluence and normalization.

MELL: Multiplicative Exponential Linear Logic

Let $\mathcal{A}$ be a set of *atomic formulae* equipped with an involutive ‡ function $\perp : \mathcal{A} \to \mathcal{A}$, called *linear negation.*

The set of formulae of the multiplicative exponential fragment of linear logic (called MELL) is defined by the following grammar, where $a \in \mathcal{A}$:

$$\mathcal{F} \ ::= \ a \mid \mathcal{F} \otimes \mathcal{F} \text{ (tensor)} \mid \mathcal{F} \parr \mathcal{F} \text{ (par)} \mid !\mathcal{F} \text{ (of course)} \mid ?\mathcal{F} \text{ (why not)}$$

We extend the notion of *linear negation* to formulae as follows:

$$\begin{array}{lcl@{\qquad\qquad}lcl} (?A)^\perp & = & !(A^\perp) & (A \otimes B)^\perp & = & A^\perp \parr B^\perp \\ (!A)^\perp & = & ?(A^\perp) & (A \parr B)^\perp & = & A^\perp \otimes B^\perp \end{array}$$

The name MELL comes from the connectors $\otimes$ and $\parr$ which are called "multiplicatives", while $!$ and $?$ are called "exponentials". While we refer the interested reader to (Girard 1987) for more details on linear logic, we give here a one-sided presentation of the sequent calculus for MELL:

$$\frac{}{\vdash A, A^\perp}\ Axiom \qquad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ Cut \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ Dereliction \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ Contraction$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ Par \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Gamma'}{\vdash \Gamma, A \otimes B, \Gamma'}\ Times \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ Weakening \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ Box$$

MELL proof nets

To all sequent derivations in MELL it is possible to associate an object called a "proof net", which allows to abstract from many inessential details in a derivation, like the order of application of independent logical rules: for example, there are many inessentially different ways to obtain $\vdash A_1 \parr A_2, \ldots, A_{n-1} \parr A_n$ from $\vdash A_1, \ldots A_n$, while there is only one proof net representing all these derivations.

Proof nets are defined inductively by rules that follow closely the ones of the one-sided sequent calculus; they are given in Figure 2. The set of proof nets is denoted PN. To simplify the drawing of a proof net, we use the following notation: a conclusion with a capital greek letter $\Gamma, \Delta, \ldots$ really stands for a set of conclusions, each one with its own wire.

Each box has exactly one conclusion preceded by a $!$, which is named "principal" port (or formula), while the other conclusions are named "auxiliary" ports (or formulae). In what follows, we will sometimes write an axiom link as $\overline{A \qquad A^\perp}$.

Reduction of proof nets

Proof nets are the "computational object" behind linear logic, because there is a notion of reduction on them (called also "cut elimination") that corresponds to the cut-elimination

‡ A function f is involutive iff $f(f(p)) = p$

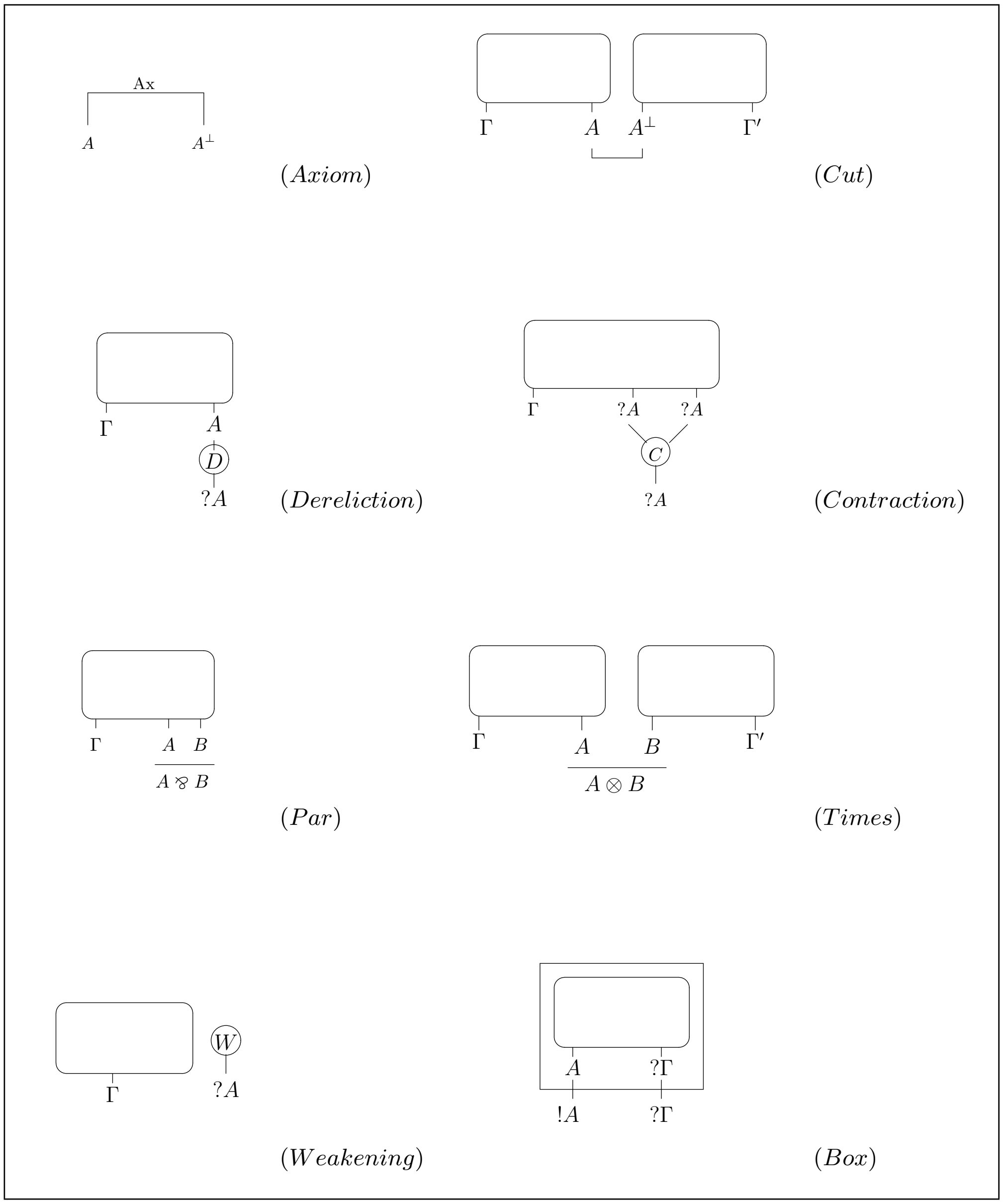
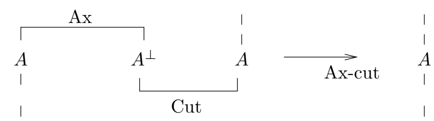


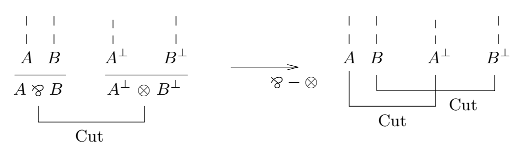
Fig. 1. MELL Proof Nets

procedure on sequent derivations. The traditional reduction system for MELL is defined as follows:

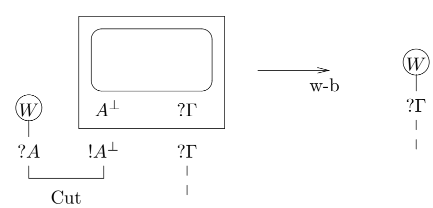
Reduction acting on a cut $Ax - cut$, removing an axiom :



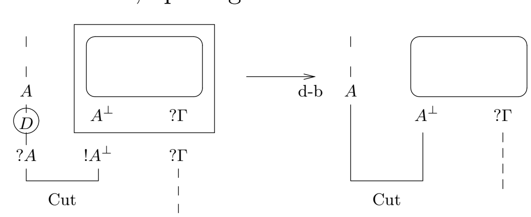
Reduction acting on a cut $⅋ - \otimes$:



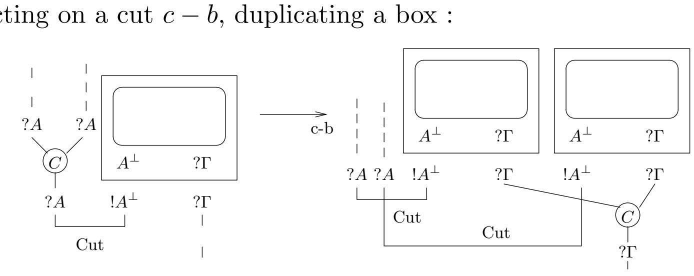
Reduction acting on a cut $w - b$, erasing a box :



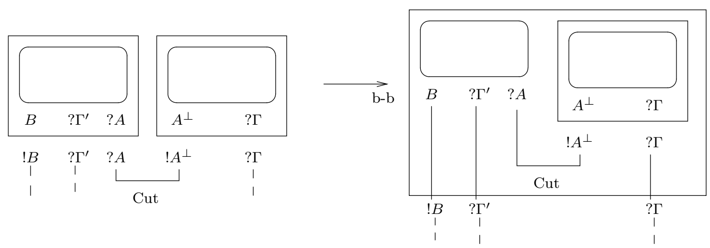
Reduction acting on a cut $d - b$, opening a box :



Reduction acting on a cut $c - b$, duplicating a box :

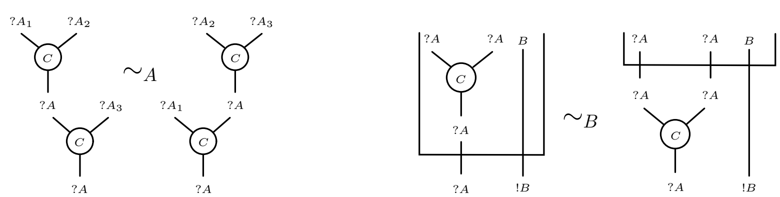


Reduction acting on a cut $b-b$, absorbing a box into another :



Extended reduction modulo an equivalence relation

Unfortunately, the original notion of reduction on PN is not well adapted to simulate neither the β rule of λ-calculus, nor the rules dealing with propagation of substitution in explicit substitution calculi: too many inessential details on the order of application of the rules are still present, and to make abstraction from them, one is naturally led to define an equivalence relation on PN, as is done in (Di Cosmo and Guerrini 1999), where the following two equivalences are introduced:



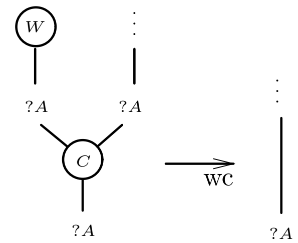
Equivalence A turns contraction into an associative operator, and corresponds to forgetting the order in which the contraction rule is used to build, for example, the derivation:

$$\dfrac{\dfrac{\vdash ?A, ?A, ?A}{\vdash ?A, ?A}\ \mathit{Contraction}}{\vdash ?A}\ \mathit{Contraction}$$

Equivalence B abstracts away the relative order of application of the rules of box-formation and contraction on the premises of a box, like in the following example.

$$\dfrac{\dfrac{\vdash ?A, ?A, B}{\vdash ?A, B}\ \mathit{Contraction}}{\vdash ?A, !B}\ \mathit{Box} \qquad \dfrac{\dfrac{\vdash ?A, ?A, B}{\vdash ?A, ?A, !B}\ \mathit{Box}}{\vdash ?A, !B}\ \mathit{Contraction}$$

Finally, besides the equivalence relation defined in (Di Cosmo and Guerrini 1999), we will also need an extra reduction rule allowing to remove unneeded weakening links when simulating explicit substitutions:



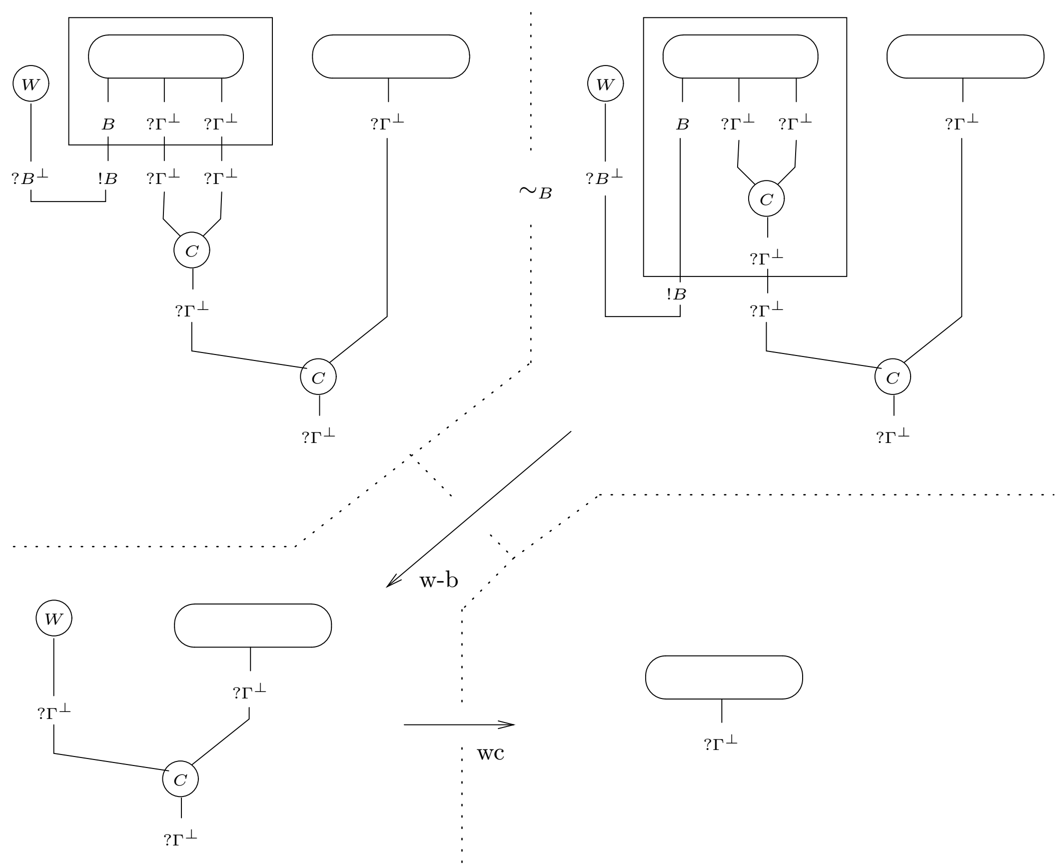
This rule allows to simplify the proof below on the left into the proof on the right

$$\cfrac{\cfrac{\cfrac{\pi}{\vdash ?A}}{\vdash ?A, ?A}\; Weakening}{\vdash ?A}\; Contraction \qquad\qquad \cfrac{\pi}{\vdash ?A}$$

Notation: We will call in the following R the system made of rules $Ax - cut$, $⅋ - \otimes$, $w-b$, $d-b$,$c-b$, $b-b$ and wc; we will name E the relation induced on PN by the contextual closure of axioms A and B; we will write R_E for the system made of the rules in R and the equivalences in E; finally, $R_E^{\neg wc}$ will stand for system R_E *without* rule *wc*.

Systems R_E and $R_E^{\neg wc}$, that contain E, are actually defining a notion of *reduction modulo an equivalence relation*, so we write for example $r \longrightarrow_{R_E} s$ if and only if there exist r' and s' such that $r =_E r' \longrightarrow_R s' =_E s$, where the equality $=_E$ is the reflexive, symmetric and transitive closure of the relation defined by A and B.

An example of reduction in R_E is given here:



The reduction R_E is flexible enough to allow an elegant simulation of β reduction and of explicit substitutions, but for that, we first need to establish that R_E is strongly normalizing. Let us see this property in the next section.

3. Termination of R_E

We know from (Di Cosmo and Guerrini 1999) that $R_E^{\neg wc}$ is terminating, and we can show easily that wc is terminating too. In this section we show that the wc-rule can be postponed with respect to all the other rules of $R_E^{\neg wc}$, so that terminationof R_E will follow from a well-known abstract lemma.

Let us first remind the following result from (Di Cosmo and Guerrini 1999):

Lemma 3.1 (Termination of $R_E^{\neg wc}$) The relation $\longrightarrow_{R_E^{\neg wc}}$ is terminating on PN.

Then, we establish the termination of wc.

Lemma 3.2 (Termination of wc) The relation $\longrightarrow_{wc}$ is terminating on PN.

Proof. The wc-rule strictly decreases the number of nodes in a proof net so no infinite wc-reduction sequence is possible. □

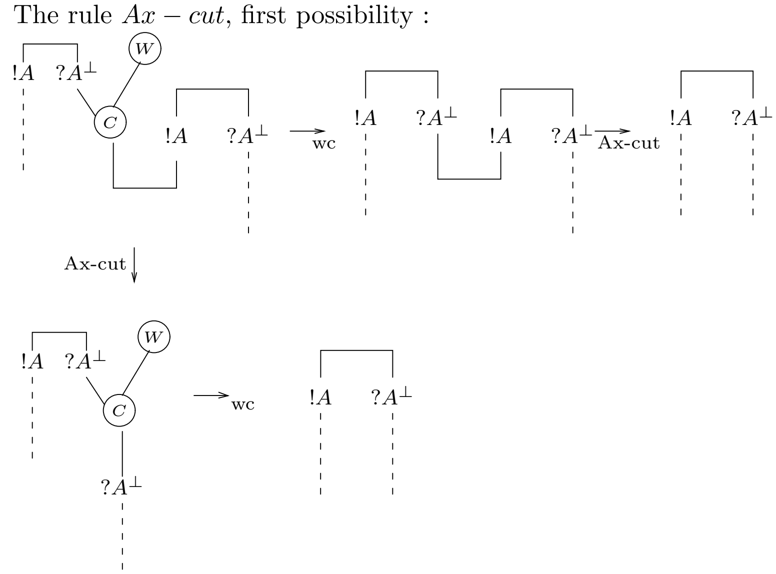
Finally, we show that given any proof net, the wc-rule can be postponed with respect to any rule of $R_E^{\neg wc}$.

Lemma 3.3 (Postponement of wc w.r.t $R_E^{\neg wc}$) Let t be a proof net. If $t \longrightarrow_{wc} \longrightarrow_{R_E^{\neg wc}} t'$, then, there is a sequence $t \longrightarrow^+_{R_E^{\neg wc}} \longrightarrow^*_{wc} t'$.

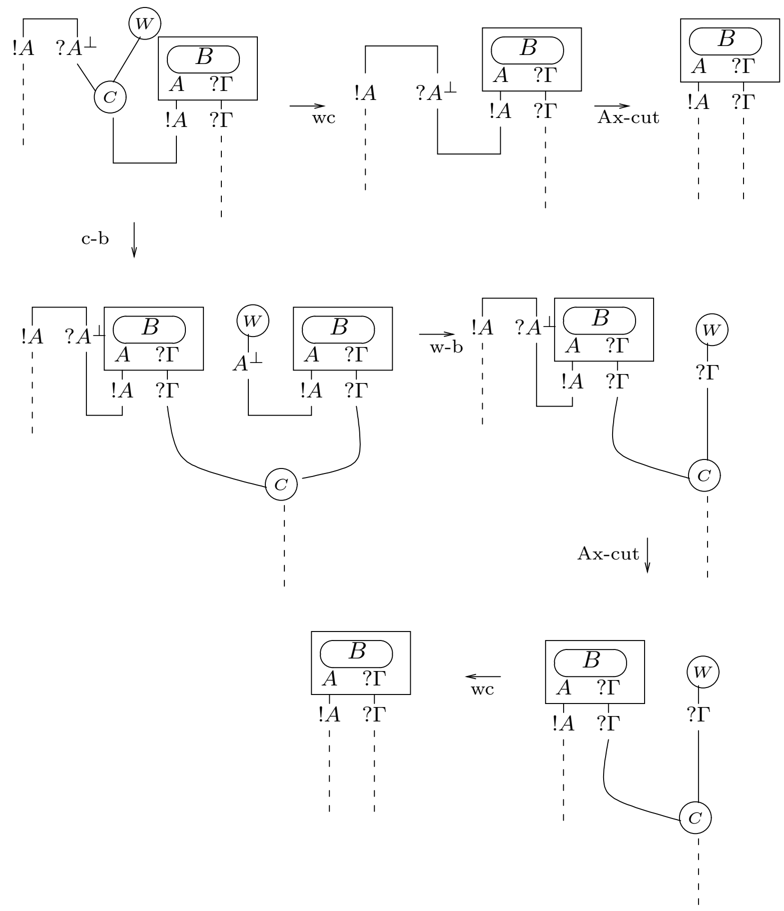
Proof. Let $t \longrightarrow_{wc} \longrightarrow_{R_E^{\neg wc}} t'$ be a reduction sequence starting at t with a wc-reduction step. Let us show that we can build an equivalent reduction $t \longrightarrow^+_{R_E^{\neg wc}} \longrightarrow^*_{wc} t'$ by analyzing all the possible cases.

We do not detail here the cases of disjoint redexes: if we apply the wc- rule followed by a rule $R1$ in $R_E^{\neg wc}$ and if the redexes occur at disjoint positions, then it is evident that $R1$ can be applied first, followed by wc, and getting the same result. We study now all the remaining cases:

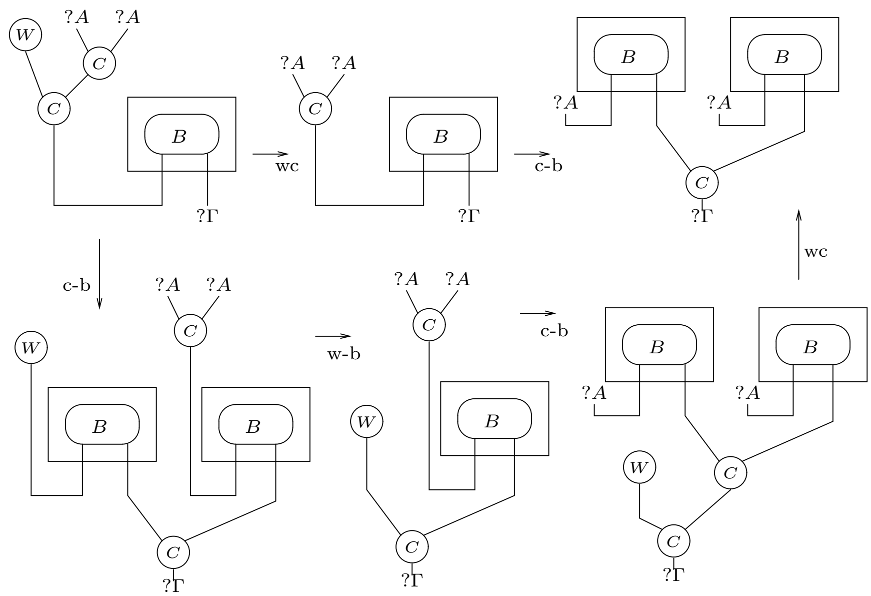
1 The rule $Ax - cut$, first possibility :



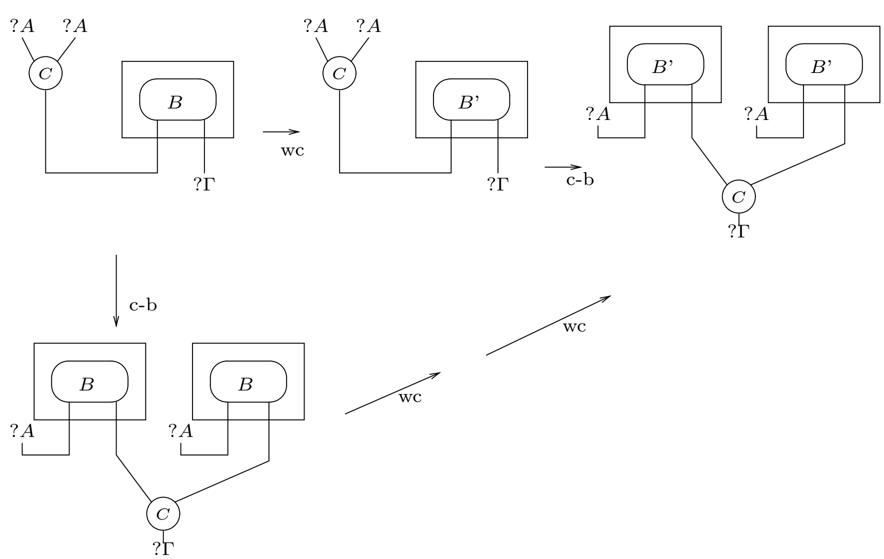
2 The rule $Ax - cut$, second possibility :



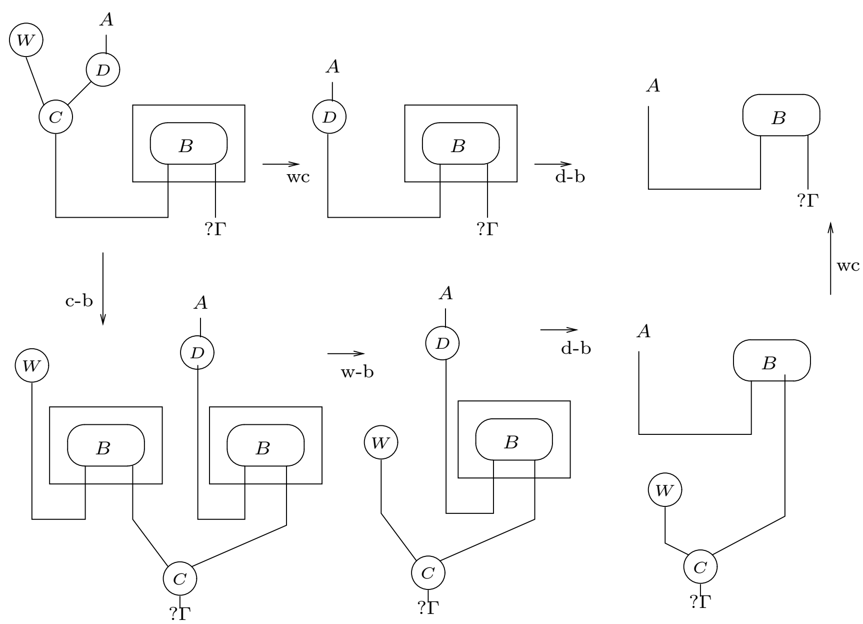
3 The rule $c-b$, first possibility :



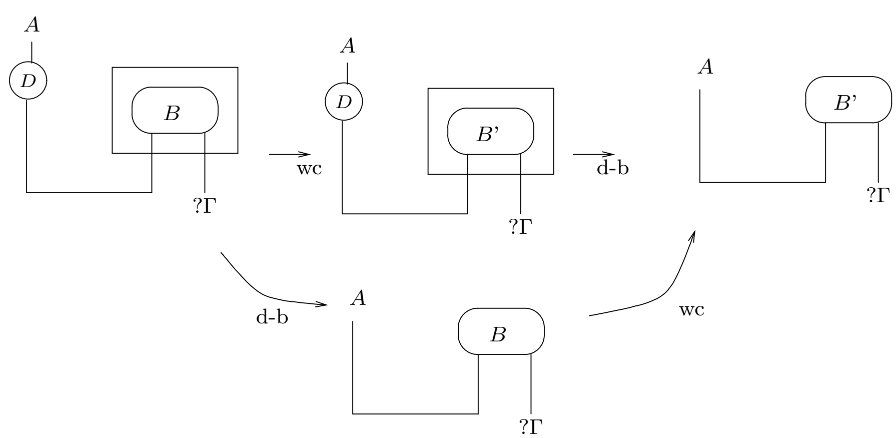
4 The rule $c-b$, second possibility :



5 The rule $d - b$, first possibility :

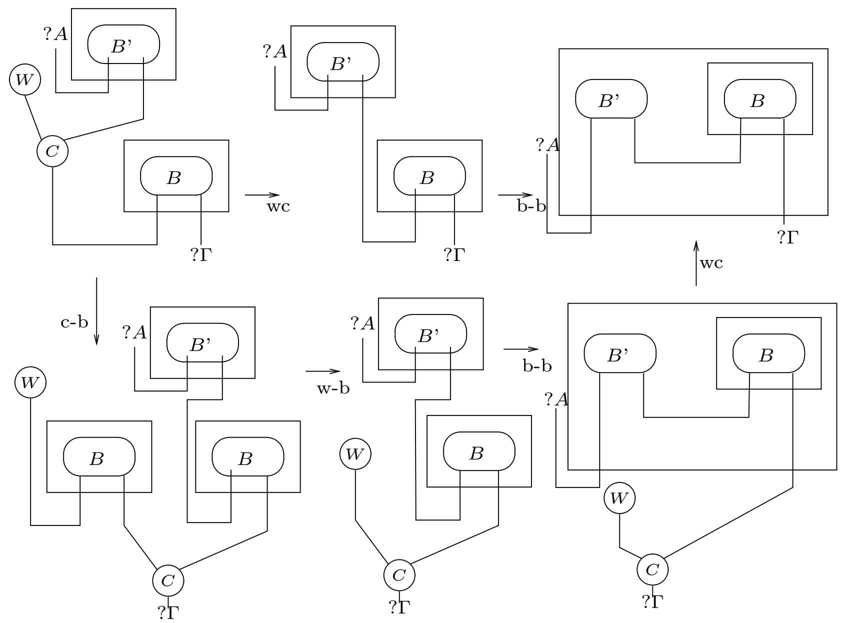


6 The rule $d - b$, second possibility :



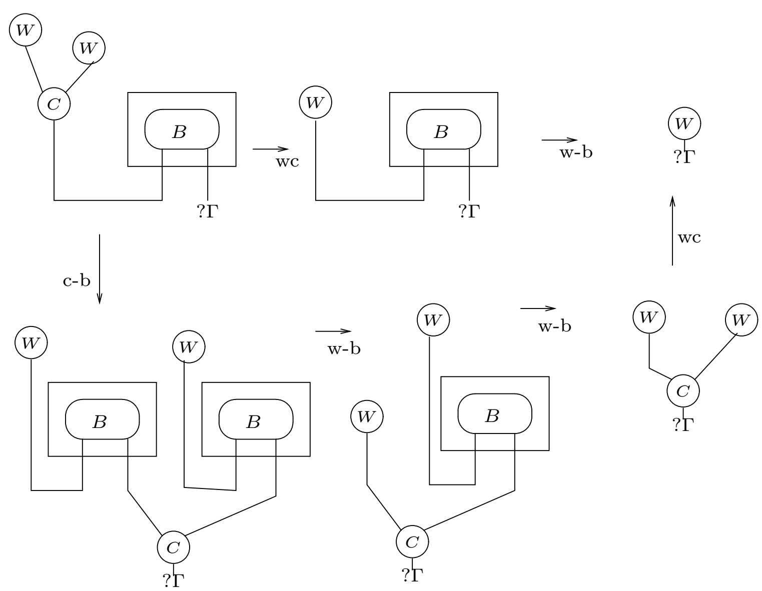
We notice that everything is happening as if the redexes were disjoint. This is due to the fact that the $d-b$ rule is non-duplicating and non-erasing w.r.t boxes. As a consequence, the *wc*-redex is still preserved after the application of the $d - b$ rule.

7 The rule $b-b$, first possibility :

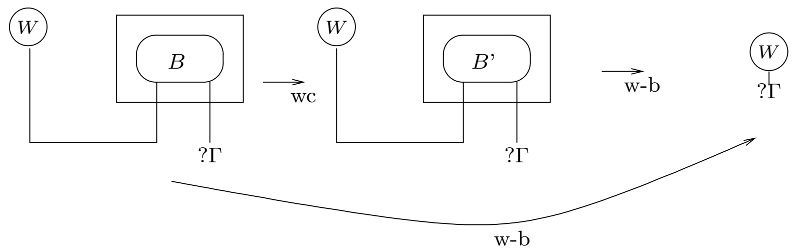


8 The rule $b-b$, second possibility : For the same reason as for $d-b$, the redexes are considered as disjoint.

9 The rule $w-b$, first possibility :



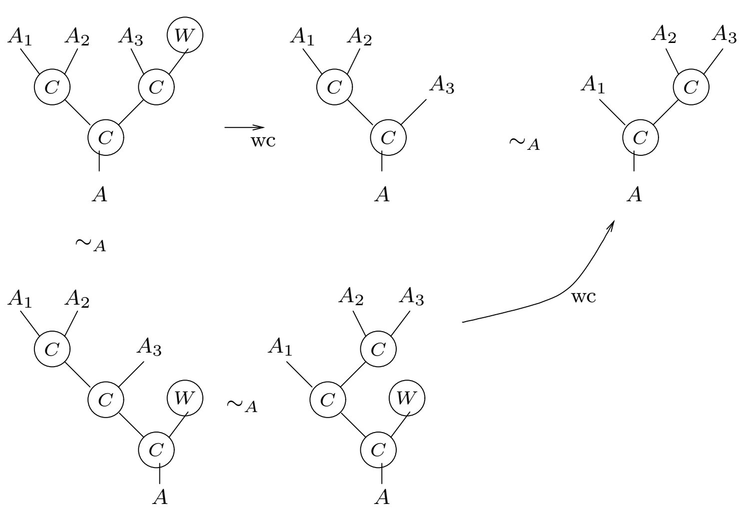
10 The rule $w - b$, second possibility :



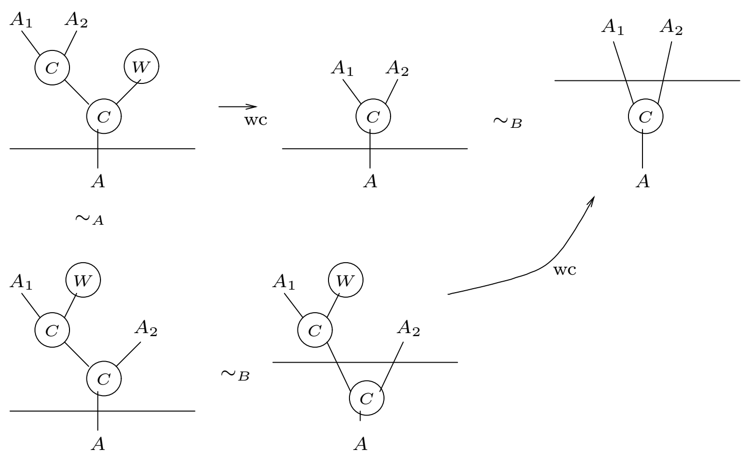
11 The rules ⅋-⊗ cut : we just notice that in this case the redexes are disjoint.

Until now we have only worked with reduction rules of R_E, but to complete our statement we also need to show that the *wc*-rule can be delayed w.r.t one equivalence step. We proceed as we did for the reduction rules. We do not study the cases where redexes are disjoint because they are evident. The remaining cases are the following:

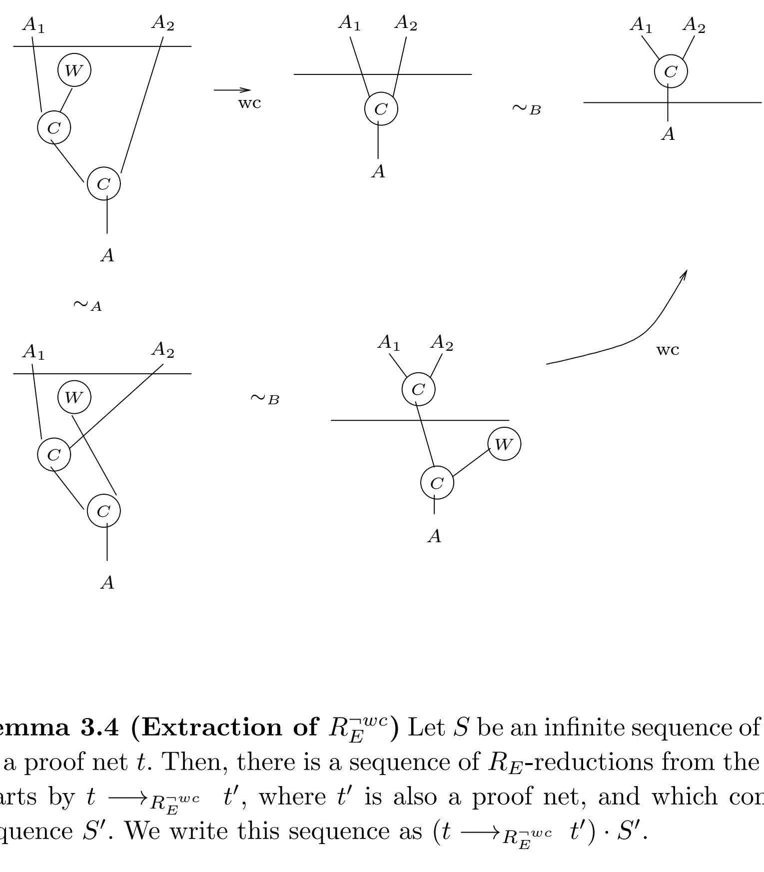
1 Associativity :



2 Box passing, first case :



3 Box passing second case :



□

Lemma 3.4 (Extraction of $R_E^{\neg wc}$) Let S be an infinite sequence of R_E-reductions starting at a proof net t. Then, there is a sequence of R_E-reductions from the same proof net t which starts by $t \longrightarrow_{R_E^{\neg wc}} t'$, where t' is also a proof net, and which continues with an infinite sequence S'. We write this sequence as $(t \longrightarrow_{R_E^{\neg wc}} t') \cdot S'$.

Proof. Let S be an infinite sequence of R_E-reductions starting at t:

$$t \longrightarrow_{R_E} \ \dots \longrightarrow_{R_E} \ \dots \longrightarrow_{R_E} \ \dots$$

We know, by Lemmas 3.2 and 3.1, that the systems wc and $R_E^{\neg wc}$ are both terminating, so it is not possible to have an infinite sequence only made of wc or $R_E^{\neg wc}$. As a consequence, the infinite sequence of R_E-reductions must be an infinite alternation of non-empty finite sequences of wc and $R_E^{\neg wc}$.

Now, there are two cases: either the alternation of sequences starts with a $R_E^{\neg wc}$-reduction step, and then the result holds by taking the sequence S without its first reduction step as S'; or the alternation starts with a wc-step:

$$t \longrightarrow^+{}_{wc} \longrightarrow^+{}_{R_E^{\neg wc}} \longrightarrow^+{}_{wc} \longrightarrow^+{}_{R_E^{\neg wc}} \ldots$$

that is, written in other way

$$t \longrightarrow^+{}_{wc} \ \longrightarrow_{R_E^{\neg wc}} t'' \longrightarrow^*{}_{R_E^{\neg wc}} \longrightarrow^+{}_{wc} \longrightarrow^+{}_{R_E^{\neg wc}} \ldots$$

In this case, we consider the sub-sequence $P = t \longrightarrow^+{}_{wc} \ \longrightarrow_{R_E^{\neg wc}} t''$ of the sequence S starting at t. This sub-sequence is composed by k reduction steps of wc and one reduction of $R_E^{\neg wc}$. Let call R the remaining sub-sequence of S.

By applying Lemma 3.3 k times on P, we can move the rule of $R_E^{\neg wc}$ at the head of the sequence. We thus obtain a finite sequence P' which begins with a reduction $t \longrightarrow_{R_E^{\neg wc}} t'$, and ends on t''. As a consequence, $P' \cdot R$ is the infinite sequence starting by a reduction $R_E^{\neg wc}$ we were looking for. □

Now it is easy to establish the fundamental theorem of this section:

Theorem 3.5 (Termination of R_E on proof nets) The reduction relation R_E is terminating on the set of proof nets.

Proof. We show it by contradiction. Let us suppose that R_E is not terminating. Then, there exist a proof net t and an infinite sequence S of R_E starting at t. By applying Lemma 3.4 to this sequence S, we obtain a sequence $(t \longrightarrow_{R_E^{\neg wc}} t') \cdot S'$ such that S' is infinite again. If we iterate this procedure an arbitrary number times, we obtain a sequence of $R_E^{\neg wc}$-reduction steps arbitrary long. This contradicts the fact that $R_E^{\neg wc}$ is terminating. □

4. From λ_{ws} with de Bruijn indices to PN

We now study the translation from typed terms of the λ_{ws}-calculus (David and Guillaume 1999; David and Guillaume 2001) into proof nets. We start by introducing the calculus, then we give the translation of types of λ_{ws} into formulae of linear logic, and the translation of terms of λ_{ws} into linear logic proof nets PN. We verify that we can correctly simulate every reduction step of λ_{ws} via the notion of reduction R_E. Finally, we use this simulation result to show strong normalization of the λ_{ws}-calculus.

4.1. *The λ_{ws}-calculus*

The λ_{ws}-calculus is a calculus with explicit substitutions where substitutions are unary (and not multiple). The version studied in this section has variables encoded with de Bruijn indices. The terms of λ_{ws} are given by the following grammar:

$M ::=$		$\underline{n}$	*variable*
	$\mid$	λM	*abstraction*
	$\mid$	(MM)	*application*
	$\mid$	$\langle k \rangle M$	*label*
	$\mid$	$[i/M, j]M$	*substitution*

Intuitively, the term $\langle k\rangle M$ means that the $k-1$ first indices in M are not "free" (in the sense of free variables of calculus with indices). The term $[i/N,j]M$ means that the $i-1$ first indices are not free in N and the $j-1$ following indices are not free in M. Those indices are used to split the typing environment of $[i/N,j]M$ in three parts: the first (resp. second) one for free variables of M (resp. N), the third one for the free variables in M and N.

The de Bruijn indices we use start with $\underline{0}$ instead of $\underline{1}$. For example, the identity function is written as $I = \lambda\underline{0}$.

The reduction rules of λ_{ws} are given in Figure 2 and the typing rules of λ_{ws} are given in Figure 3, where we suppose that $|\Gamma| = i$ and $|\Delta| = j$.

$$
\begin{array}{rrcll}
(b_1) & (\lambda M N) & \longrightarrow & [0/N,0]M & \\
(b_2) & (\langle k\rangle(\lambda M) N) & \longrightarrow & [0/N,k]M & \\
(f) & [i/N,j](\lambda M) & \longrightarrow & \lambda[i+1/N,j]M & \\
(a) & [i/N,j](MP) & \longrightarrow & ([i/N,j]M)([i/N,j]P) & \\
(e_1) & [i/N,j]\langle k\rangle M & \longrightarrow & \langle j+k-1\rangle M & if\ i<k \\
(e_2) & [i/N,j]\langle k\rangle M & \longrightarrow & \langle k\rangle[i-k/N,j]M & if\ i\geq k \\
(n_1) & [i/N,j]\underline{k} & \longrightarrow & \underline{k} & if\ i>k \\
(n_2) & [i/N,j]\underline{i} & \longrightarrow & \langle i\rangle N & \\
(n_3) & [i/N,j]\underline{k} & \longrightarrow & \underline{j+k-1} & if\ i<k \\
(c_1) & [i/N,j][k/P,l]M & \longrightarrow & [k/[i-k/N,j]P,j+l-1]M & if\ k\leq i<k+l \\
(c_2) & [i/N,j][k/P,l]M & \longrightarrow & [k/[i-k/N,j]P,l][i-l+1/N,j]M & if\ i\geq k+l \\
(d) & \langle i\rangle\langle j\rangle M & \longrightarrow & \langle i+j\rangle M &
\end{array}
$$

Fig. 2. Reduction rules of λ_{ws} with de Bruijn indices

$$
\frac{}{\Gamma, A, \Delta \vdash \underline{i} : A}\ Ax
\qquad
\frac{\Delta \vdash M : B}{\Gamma, \Delta \vdash \langle i\rangle M : B}\ Weak
$$

$$
\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash (MN) : A}\ App
\qquad
\frac{B, \Gamma \vdash M : C}{\Gamma \vdash \lambda M : B \to C}\ Lamb
$$

$$
\frac{\Delta, \Pi \vdash N : A \quad \Gamma, A, \Pi \vdash M : B}{\Gamma, \Delta, \Pi \vdash [i/N,j]M : B}\ Sub
$$

Fig. 3. Typing rules for λ_{ws} with de Bruijn indices

We notice that for each well-typed term of the λ_{ws}-calculus, there is only one possible typing judgment. This will simplify the proof of simulation of λ_{ws} by easily considering the unique typing judgment of terms.

As expected the λ_{ws}-calculus enjoys the subject reduction property (Guillaume 1999).

Theorem 4.1 (Subject Reduction) If $\Psi \vdash M : C$ and $M \longrightarrow M'$, then $\Psi \vdash M' : C$.

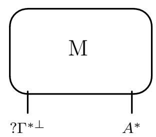
4.2. *Translation of types and terms of λ_{ws}*

We use the translation of types introduced in (Danos, Joinet and Schellinx 1995) given by :

$$\begin{array}{lcll} A^* & = & A & \text{if } A \text{ is an atomic type} \\ (A \to B)^* & = & ?((A^*)^\perp) \parr !B^* & \text{(that is, } !A^* \multimap !B^*\text{) otherwise} \end{array}$$

Since wires are commutative in proof nets, we feel free to exchange them when we define the translation of a term. The translation associates to every typed term M of λ_{ws}, whose typing judgment ends with the conclusion written below on the left, a proof net having the shape sketched below on the right:

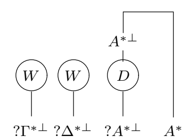
$$\overline{\Gamma \vdash M : A}$$



Here is the formal definition of the translation T from λ_{ws}-terms into proof nets.

— If the term is a variable and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

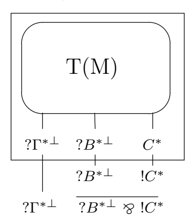
$$\overline{\Gamma, A, \Delta \vdash \underline{i} : A}\ Ax$$



where i is the position of A in the typing environment,

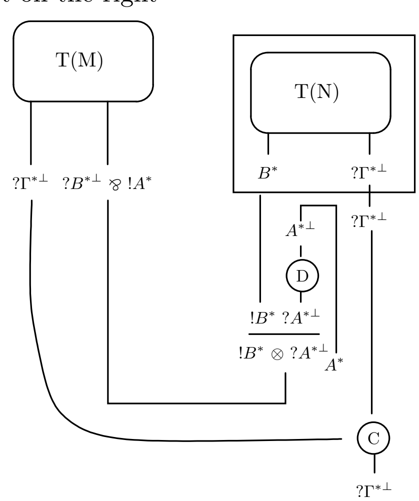
— If the term is a λ-abstraction and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\frac{B, \Gamma \vdash M : C}{\Gamma \vdash \lambda M : B \to C}\ Lamb$$



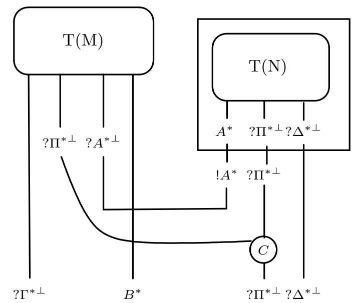
— If the term is an application and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash (MN) : A}\ App$$



— If the term is a substitution and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

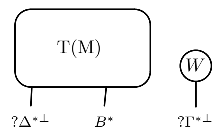
$$\frac{\Delta, \Pi \vdash N : A \quad \Gamma, A, \Pi \vdash M : B}{\Gamma, \Delta, \Pi \vdash [i/N, j]M : B}\ Sub$$



where i is the length of the list Γ and j is the length of the list Δ, then its translation is the proof net

— Finally, if the term is a label and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\frac{\Delta \vdash M : B}{\Gamma, \Delta \vdash \langle i \rangle M : B}\ Weak$$



where i is the length of the list Γ, then its translation is the proof net

4.3. *Simulating λ_{ws}-reduction*

We now verify that our notion of reduction R_E on PN simulates the λ_{ws}-reduction on typed λ_{ws}-terms. It is in this proof that we find the motivation for our choice of translation from λ-terms into proof nets: with the more traditional translation sending the intuitionistic type $A \to B$ into the linear $!A \multimap B$, the simulation of the rewrite rule f would give rise to an equality, not to a reduction step like in this paper.

Notation: In the proof of the following lemma, we will draw several complex proof nets, where the translations $?\Gamma^{*\perp}, ?\Delta^{*\perp}, ?\Pi^{*\perp}$, etc. of the environments Γ, Δ, Π, etc. appear repeated many times. In order to make these pictures more readable, we will make a slight abuse of notation, only in the following proof, by simply writing Γ in place of its correct translation $?\Gamma^{*\perp}$.

Lemma 4.2 (Simulation of λ_{ws}) The relation R_E simulates the λ_{ws}-reduction on typed terms: if $t \longrightarrow_{\lambda_{ws}} t'$, then $T(t) \longrightarrow^+{}_{R_E} T(t')$, excepted for the rules e_2 and d for which we have $T(t) = T(t')$.

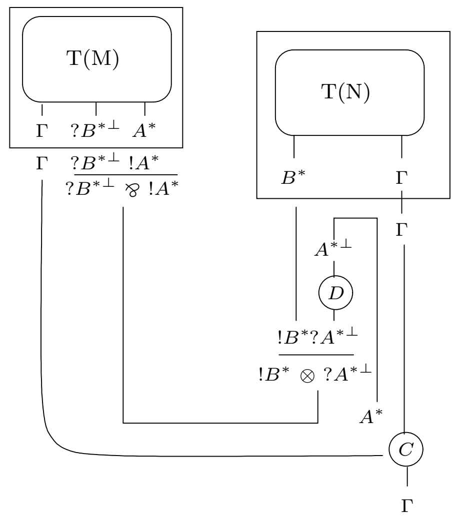
Proof. The proof proceeds by cases on the reduction rule applied in the step $t \longrightarrow_{\lambda_{ws}} t'$. Since reductions λ_{ws} and R_E are closed under all contexts, we only need to study the cases where reduction takes place at the head position of t. In the proof, rule wc is used to simulate b_2, e_1, n_1, n_2, n_3, equivalence A is used to simulate a, c_1, c_2, and equivalence B is used to simulate f, a, c_1, c_2.

— **Rule** b_1 **:** $(\lambda M N) \longrightarrow\ [0/N, 0]M$

The typing judgment of $(\lambda M N)$ ends with

$$\cfrac{\cfrac{B, \Gamma \vdash M : A}{\Gamma \vdash \lambda M : B \to A}\ Lamb \qquad \Gamma \vdash N : B}{\Gamma \vdash ((\lambda M) N) : A}\ App$$

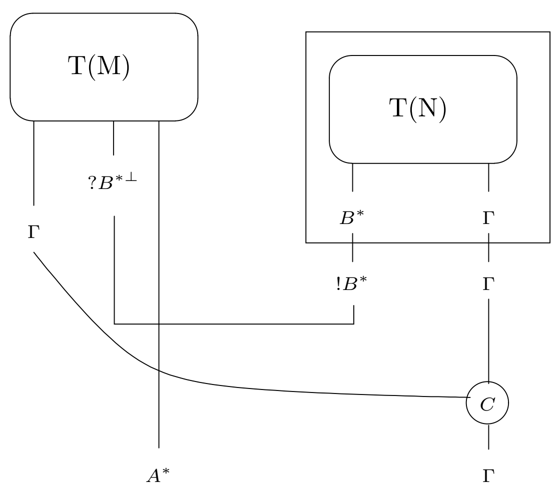
and its translation is the proof net



The typing judgment of $[0/N,0]M$ must end with:

$$\frac{B,\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash [0/N,0]M : A}\ Sub$$

and its translation is the proof net



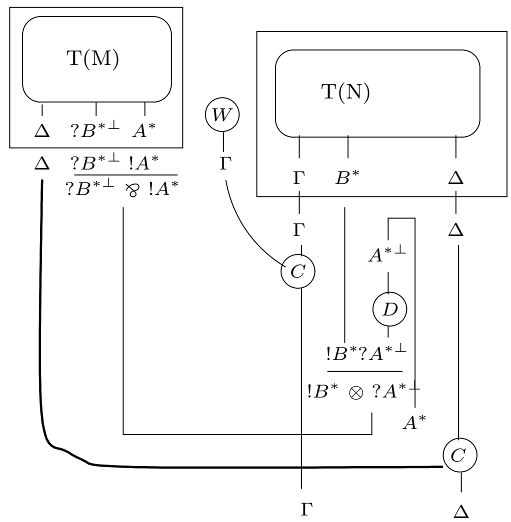
Starting from the first proof net, we eliminate the ⅋-$\otimes$ cut, then the $d-b$ cut and finally the $Ax-cut$ cut to obtain the final proof net.

— **Rule** b_2 **:** $((\langle k\rangle\lambda M)N) \longrightarrow\ [0/N,k]M$

The typing environment can be split in two parts Γ and Δ, where k is the length of Γ. The typing judgment of $((\langle k\rangle\lambda M)N)$ ends with

$$\frac{\dfrac{\dfrac{B,\Delta \vdash M : A}{\Delta \vdash \lambda M : B \to A}}{\Gamma,\Delta \vdash \langle k\rangle\lambda M : B \to A} \quad \Gamma,\Delta \vdash N : B}{\Gamma,\Delta \vdash ((\langle k\rangle\lambda M)N) : A}$$

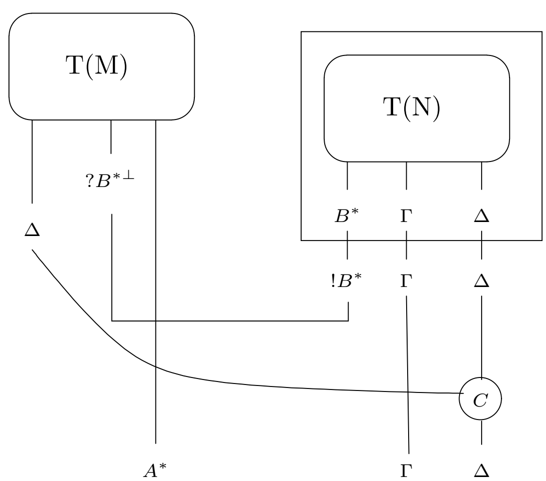
and its translation is the proof net



The typing judgment of $[0/N, k]M$ must end with:

$$\frac{\Gamma, \Delta \vdash N : B \quad B, \Delta \vdash M : A}{\Gamma, \Delta \vdash [0/N, k]M : A}\ Sub$$

and its translation is the proof net

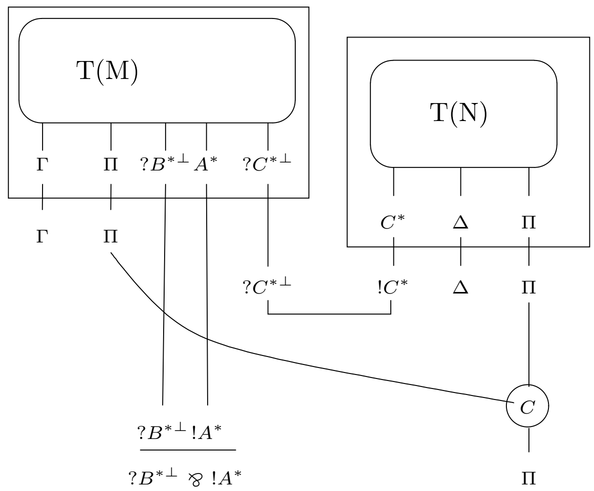


As for the b_1 rule, we eliminate the $⅋$-$\otimes$ cut, then the $d - b$ cut, and the $Ax - cut$ cut. Finally, we apply the wc rule to achieve the desired result.

— **Rule** f **:** $[i/N, j]\lambda M \longrightarrow \lambda[i + 1/N, j]M$
The typing environment can be split in three parts Γ, Δ, Π, where i is the length of Γ and j is the length of Δ. The typing judgment of $[i/N, j]\lambda M$ ends with

$$\dfrac{\Delta, \Pi \vdash N : C \quad \dfrac{B, \Gamma, C, \Pi \vdash M : A}{\Gamma, C, \Pi \vdash \lambda M : B \to A}\ Lamb}{\Gamma, \Delta, \Pi \vdash [i/N, j]\lambda M : B \to A}\ Sub$$

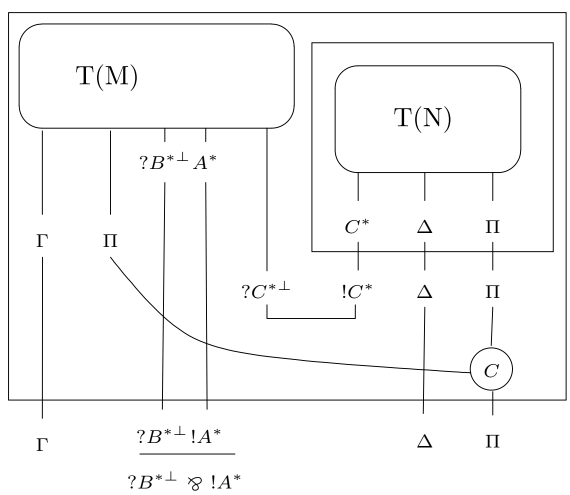
and its translation is the proof net



The typing judgment of $\lambda[i+1/N,j]M$ must end with:

$$\dfrac{\dfrac{\Delta,\Pi \vdash N : C \quad B,\Gamma,C,\Pi \vdash M : A}{B,\Gamma,\Delta,\Pi \vdash [i+1/N,j]M : A}\,Sub}{\Gamma,\Delta,\Pi \vdash \lambda[i+1/N,j]M : B \to A}\,Lamb$$

and its translation is the proof net

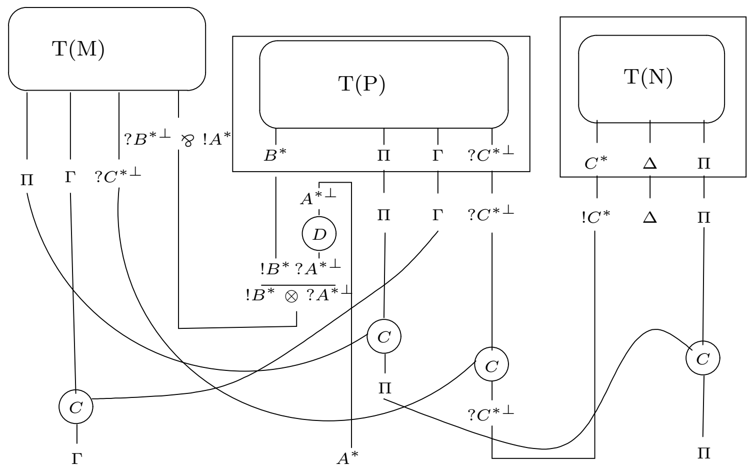


To reduce the first proof net into the second one, we must eliminate the $b-b$ cut, then use the equivalence relation B (we will exactly show how to use the equivalence relations in the case of the rule a).

— **Rule** a **:** $[i/N,j](MP) \longrightarrow (([i/N,j]M)([i/N,j]P))$
The typing environment can be split in three parts Γ, Δ, Π, where i is the length of Γ and j is the length of Δ. The typing judgment of $[i/N,j](MP)$ ends with

$$\dfrac{\Delta,\Pi \vdash N : C \quad \dfrac{\Gamma,C,\Pi \vdash M : B \to A \quad \Gamma,C,\Pi \vdash P : B}{\Gamma,C,\Pi \vdash MP : A}\,App}{\Gamma,\Delta,\Pi \vdash [i/N,j](MP) : A}\,Sub$$

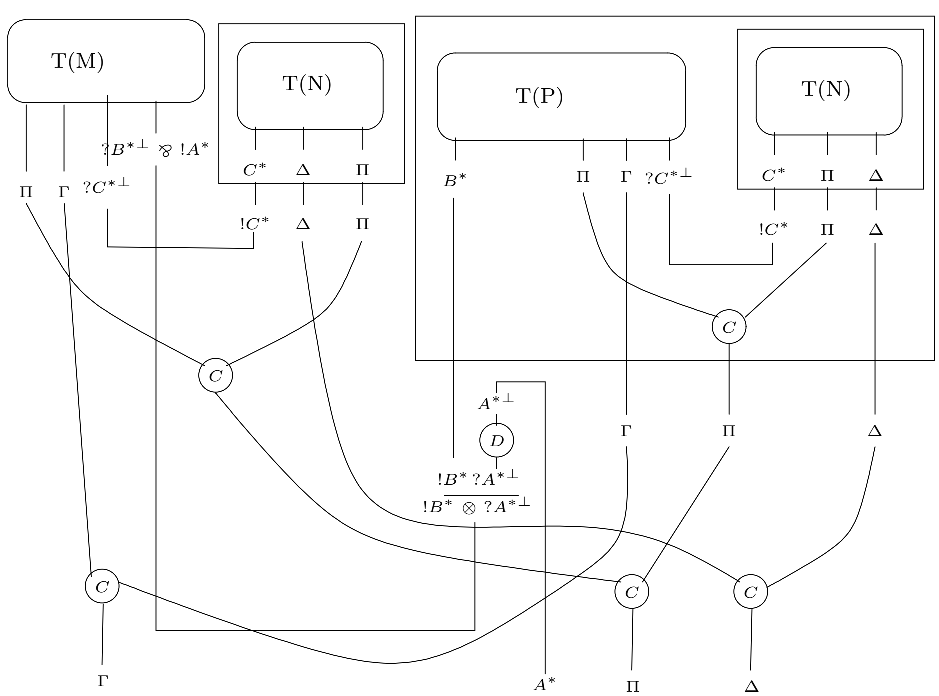
and its translation is the proof net



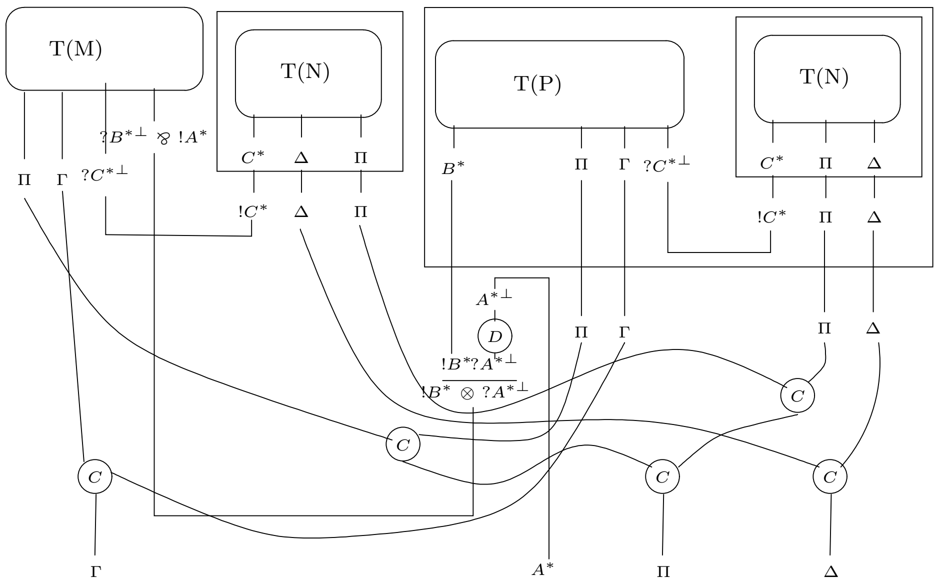
The typing judgment of $(([i/N,j]M)([i/N,j]P))$ must end with:

$$\dfrac{\dfrac{\Delta,\Pi\vdash N:C \quad \Gamma,C,\Pi\vdash M:B\to A}{\Gamma,\Delta,\Pi\vdash([i/N,j]M):B\to A}\,Sub \qquad \dfrac{\Delta,\Pi\vdash N:C \quad \Gamma,C,\Pi\vdash P:B}{\Gamma,\Delta,\Pi\vdash([i/N,j]P):B}\,Sub}{\Gamma,\Delta,\Pi\vdash(([i/N,j]M)([i/N,j]P)):A}\,App$$

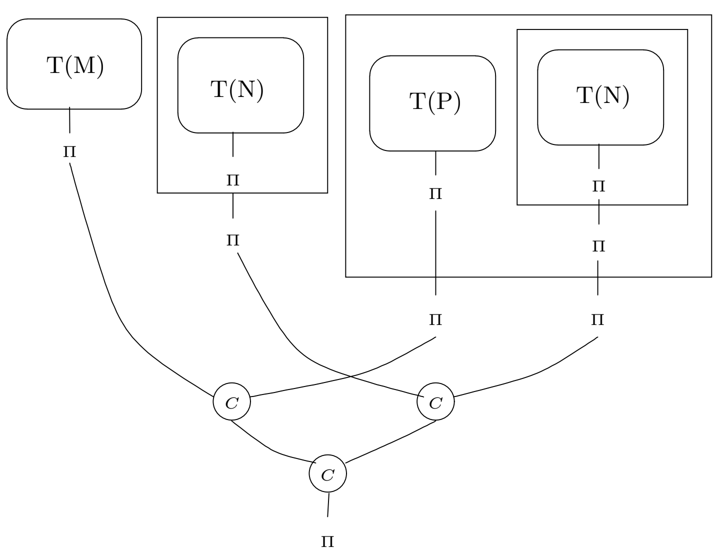
and its translation is the proof net



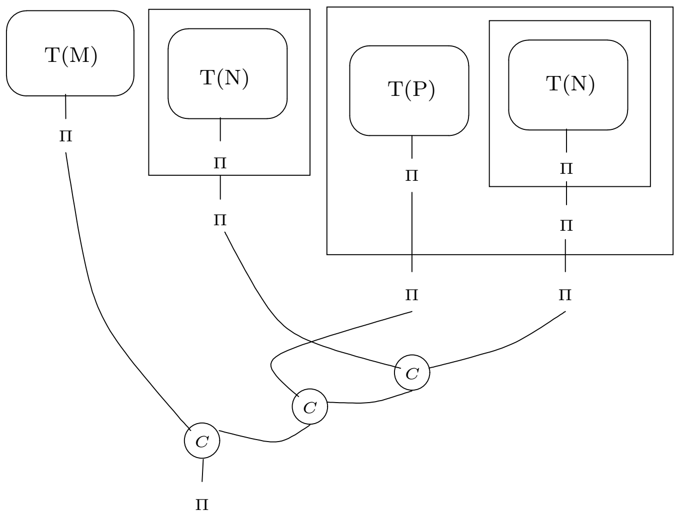
We eliminate the $c-b$ cut, then the $b-b$ cut, and thus we get the following proof net:



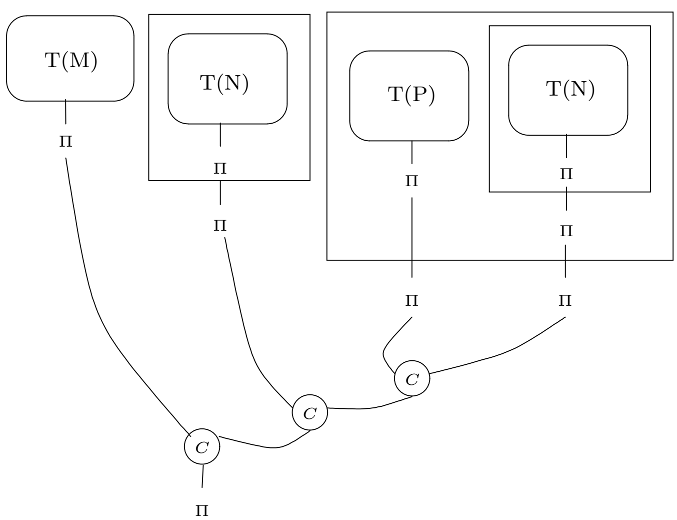
To get to the desired proof net we need to use the equivalence relations A and B which were introduced in Section 2. To better understand how to use them, we focus on the crucial informations, i.e. the contraction nodes and their connections with the nets $T(M)$, $T(N)$ and $T(P)$. Here is the net corresponding to the above net :



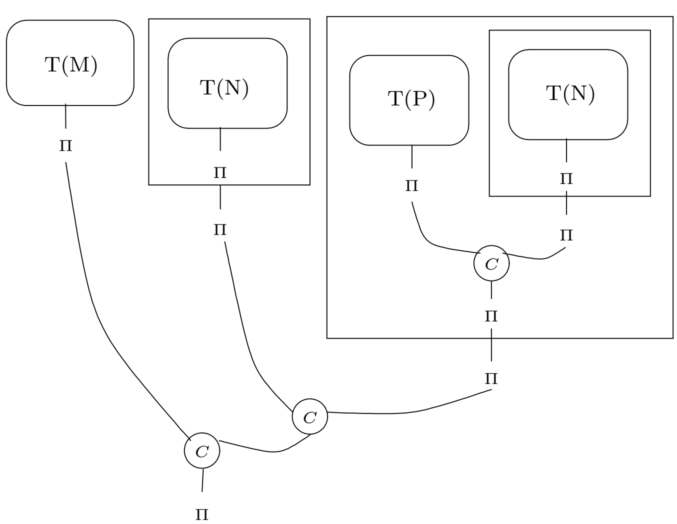
We use the associativity axiom A to obtain :



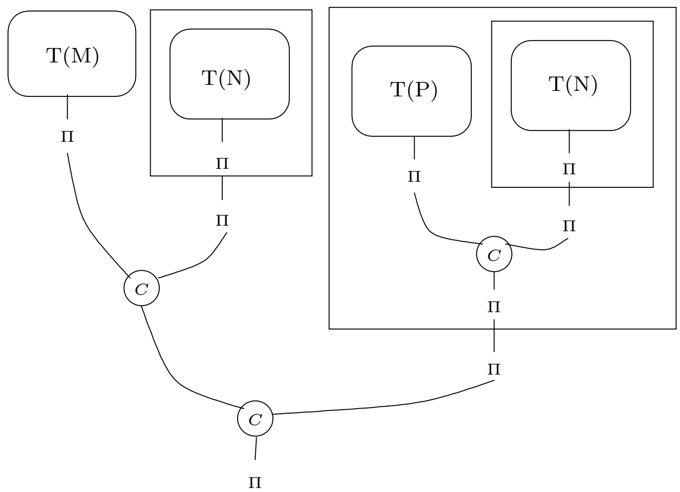
Again by associativity we get



Using the B axiom we can put the contraction inside the box :



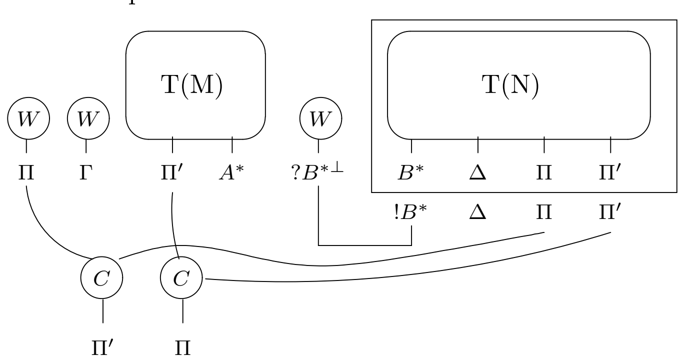
And finally we use the A axiom again to obtain the desired proof net :



— **Rule** e_1 **:** $[i/N, j]\langle k\rangle M \longrightarrow \langle j+k-1\rangle M$ if $i < k$
The typing environment can be split in four parts Γ, Δ, Π, and Π', where i is the length of Γ, j is the length of Δ, and k ($k > i$) is the length of Γ plus the length of Π plus 1. The typing judgment of $[i/N, j]\langle k\rangle M$ ends with

$$\dfrac{\Delta, \Pi, \Pi' \vdash N : B \quad \dfrac{\Pi' \vdash M : A}{\Gamma, B, \Pi, \Pi' \vdash \langle k\rangle M : A}\,Weak}{\Gamma, \Delta, \Pi, \Pi' \vdash [i/N, j]\langle k\rangle M : A}\,Sub$$

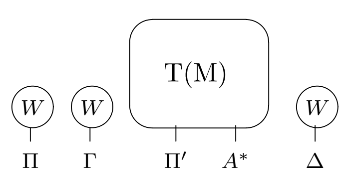
and its translation is the proof net



The typing judgment of $\langle j+k-1\rangle M$ must end with:

$$\dfrac{\Pi' \vdash M : A}{\Gamma, \Delta, \Pi, \Pi' \vdash \langle j+k-1\rangle M : A}\,Weak$$

and its translation is the net

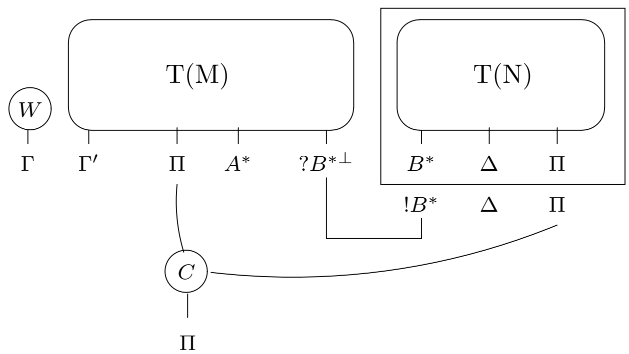


Starting from the first proof net, we eliminate the $w - b$ cut, then we apply the wc rule and we finally obtain the desired proof net.

— **Rule** e_2 **:** $[i/N,j]\langle k\rangle M \longrightarrow \langle k\rangle[i-k/N,j]M$ if $i \geq k$
The typing environment can be split in four parts Γ, Γ', Δ, Π, where i is the length of Γ plus the length of Γ', j is the length of Δ and k ($k \leq i$) is the length of Γ. The typing judgment of $[i/N,j]\langle k\rangle M$ ends with

$$\dfrac{\Delta,\Pi \vdash N:B \quad \dfrac{\Gamma',B,\Pi \vdash M:A}{\Gamma,\Gamma',B,\Pi \vdash \langle k\rangle M:A}\, Weak}{\Gamma,\Gamma',\Delta,\Pi \vdash [i/N,j]\langle k\rangle M:A}\, Sub$$

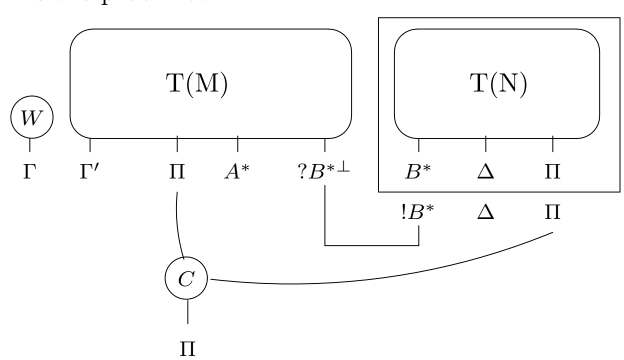
and its translation is the proof net



The typing judgment of $\langle k\rangle[i-k/N,j]M$ must end with:

$$\dfrac{\dfrac{\Delta,\Pi \vdash N:B \quad \Gamma',B,\Pi \vdash M:A}{\Gamma',\Delta,\Pi \vdash [i-k/N,j]M:A}\, Sub}{\Gamma,\Gamma',\Delta,\Pi \vdash \langle k\rangle[i-k/N,j]M:A}\, Weak$$

and its translation is the proof net

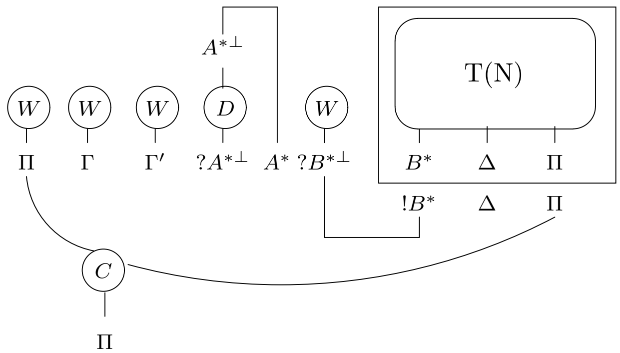


We notice that the two nets are already the same. This is the first of the exception cases of the lemma.

— **Rule** n_1 **:** $[i/N,j]\underline{k} \longrightarrow \underline{k}$ if $i > k$
The typing environment can be split in five parts Γ, A, Γ', Δ, Π, where i is the length of Γ plus the length of Γ' plus 1, j is the length of Δ and k ($k < i$) is the length of Γ. The typing judgment of $[i/N,j]\underline{k}$ ends with

$$\dfrac{\Delta,\Pi \vdash N:B \quad \dfrac{}{\Gamma,A,\Gamma',B,\Pi \vdash \underline{k}:A}\, Ax}{\Gamma,A,\Gamma',\Delta,\Pi \vdash [i/N,j]\underline{k}:A}\, Sub$$

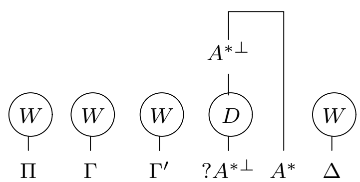
and its translation is the proof net



The typing judgment of $\underline{k}$ must end with:

$$\overline{\Gamma, A, \Gamma', \Delta, \Pi \vdash \underline{k} : A}$$

and its translation is the proof net



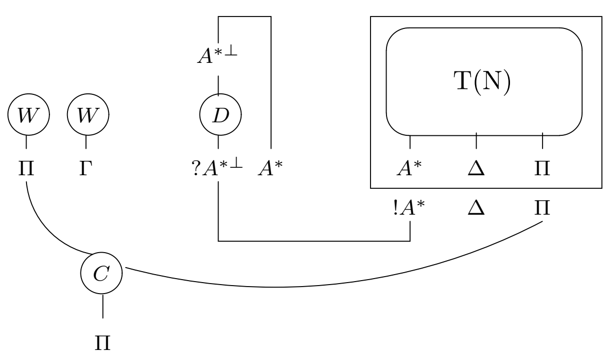
To reduce the first proof net into the second one it is enough to eliminate the $w - b$ cut and to apply the wc rule.

— **Rule** n_2 **:** $[i/N, j]\underline{i} \longrightarrow \langle i \rangle N$

The typing environment can be split in three parts Γ, Δ, Π, where i is the length of Γ and j is the length of Δ. The typing judgment of $[i/N, j]\underline{i}$ ends with

$$\dfrac{\Delta, \Pi \vdash N : A \quad \overline{\Gamma, A, \Pi \vdash \underline{i} : A}\ Ax}{\Gamma, \Delta, \Pi \vdash [i/N, j]\underline{i} : A}\ Sub$$

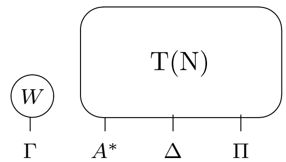
and its translation is the proof net



The typing judgment of $\langle i \rangle N$ must end with:

$$\dfrac{\Delta, \Pi \vdash N : A}{\Gamma, \Delta, \Pi \vdash \langle i \rangle N : A}\ Weak$$

and its translation is the proof net

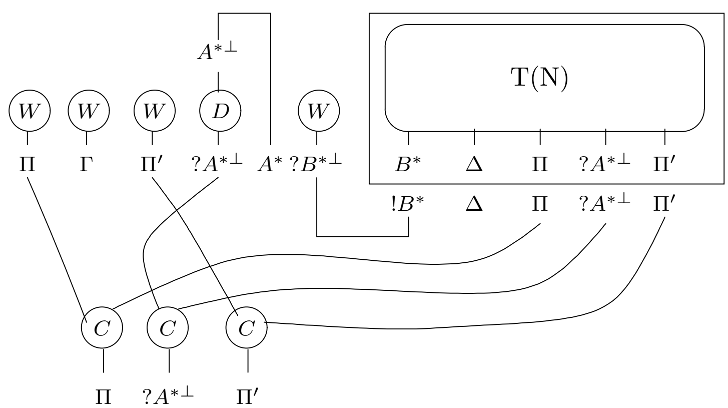


Starting from the first proof net, we eliminate the $d-b$ cut, then the $Ax-cut$ cut, and we apply the wc rule to obtain the desired proof net.

— **Rule** n_3 **:** $[i/N, j]\underline{k} \longrightarrow \underline{j+k-1}$ if $i < k$
The typing environment can be split in five parts Γ, Δ, Π, A, Π', where i is the length of Γ, j is the length of Δ and k ($k > i$) is the length of Γ plus the length of Π plus 1. The typing judgment of $[i/N, j]\underline{k}$ ends with

$$\dfrac{\Delta, \Pi, A, \Pi' \vdash N : B \quad \overline{\Gamma, B, \Pi, A, \Pi' \vdash \underline{k} : A}\ Ax}{\Gamma, \Delta, \Pi, A, \Pi' \vdash [i/N, j]\underline{k} : A}\ Sub$$

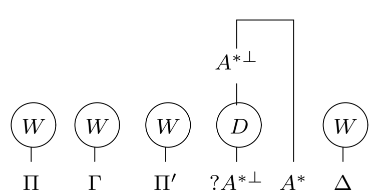
and its translation is the proof net



The typing judgment of $\underline{j+k-1}$ must end with:

$$\overline{\Gamma, \Delta, \Pi, A, \Pi' \vdash \underline{j+k-1} : A}\ Ax$$

and its translation is the proof net

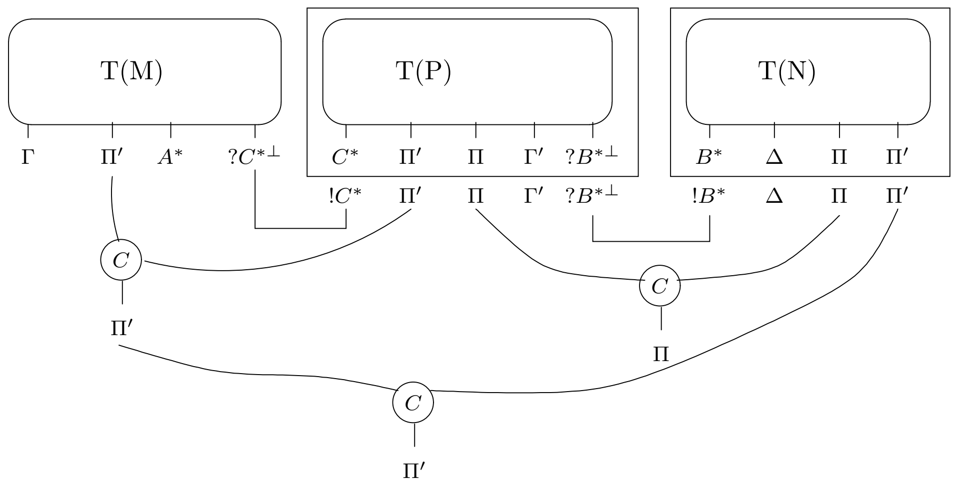


As for the n_1 rule, we eliminate the $w-b$ cut, then we apply three times the wc rule to achieve the desired result.

— **Rule** c_1 **:** $[i/N, j][k/P, l]M \longrightarrow [k/[i-k/N, j]P, j+l-1]M$ if $k \leq i < k+l$
The typing environment can be split into five parts Γ, Γ', Δ, Π, Π', where i is the length of Γ plus the length of Γ', j is the length of Δ, k ($k \leq i$) is the length of Γ and l ($k+l > i$) is the length of Γ' plus the length of Π plus 1. The typing judgment of $[i/N, j][k/P, l]M$ ends with

$$\dfrac{\Delta, \Pi, \Pi' \vdash N : B \qquad \dfrac{\Gamma', B, \Pi, \Pi' \vdash P : C \quad \Gamma, C, \Pi' \vdash M : A}{\Gamma, \Gamma', B, \Pi, \Pi' \vdash [k/P, l]M : A}\,Sub}{\Gamma, \Gamma', \Delta, \Pi, \Pi' \vdash [i/N, j][k/P, l]M : A}\,Sub$$

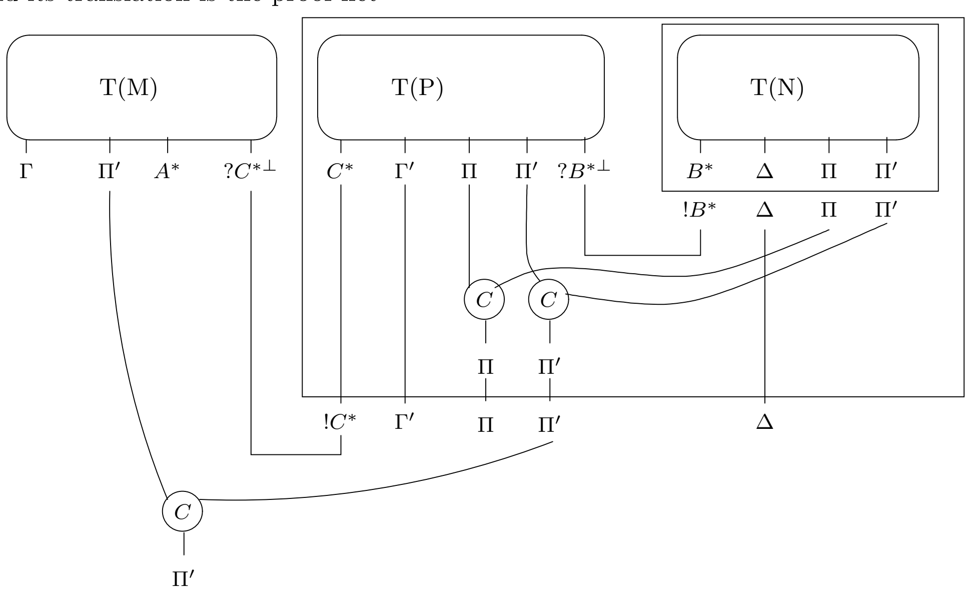
and its translation is the proof net



The typing judgment of $[k/[i-k/N,j]P, j+l-1]M$ must end with:

$$\dfrac{\dfrac{\Delta, \Pi, \Pi' \vdash N : B \quad \Gamma', B, \Pi, \Pi' \vdash P : C}{\Gamma', \Delta, \Pi, \Pi' \vdash [i-k/N, j]P : C}\,Sub \qquad \Gamma, C, \Pi' \vdash M : A}{\Gamma, \Gamma', \Delta, \Pi, \Pi' \vdash [k/[i-k/N, j]P, j+l-1]M : A}\,Sub$$

and its translation is the proof net



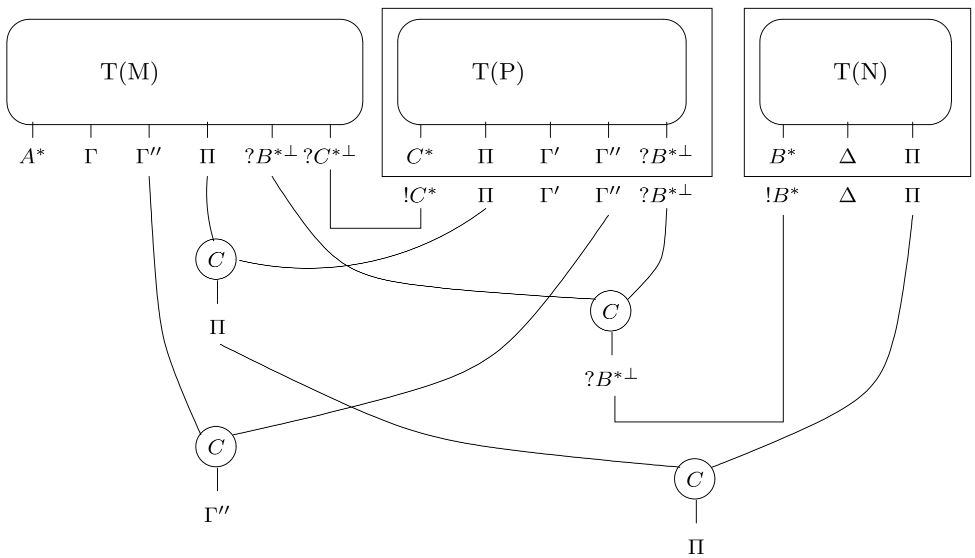
To reduce the first proof net into the second one, we must eliminate the $b-b$ cut, then apply the equivalence relations A and B.

— **Rule** c_2 **:** $[i/N, j][k/P, l]M \longrightarrow [k/[i-k/N, j]P, l][i-l+1/N, j]M$ if $k+l \leq i$
The typing environment can be split in five parts Γ, Γ', Γ'', Δ, Π, where i is the length of Γ plus the length of Γ' plus the length of Γ'', j is the length of Δ, k ($k+l \leq i$) is the

length of Γ and l is the length of Γ'. The typing judgment of $[i/N,j][k/P,l]M$ ends with

$$\dfrac{\Delta,\Pi\vdash N:B \quad \dfrac{\Gamma',\Gamma'',B,\Pi\vdash P:C \quad \Gamma,C,\Gamma'',B,\Pi\vdash M:A}{\Gamma,\Gamma',\Gamma'',B,\Pi\vdash [k/P,l]M:A}\,Sub}{\Gamma,\Gamma',\Gamma'',\Delta,\Pi\vdash [i/N,j][k/P,l]M:A}\,Sub$$

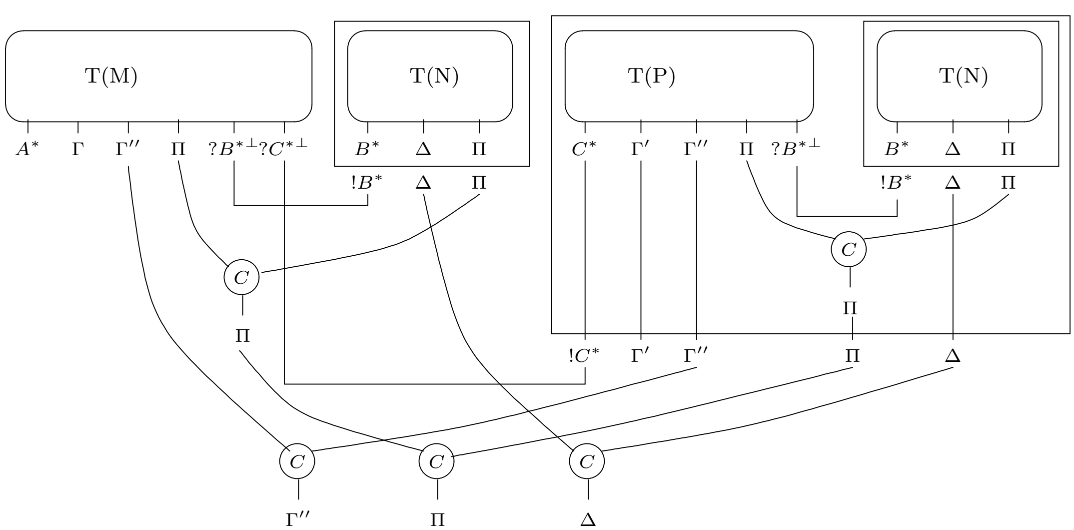
and its translation is the proof net



The typing judgment of $[k/[i-k/N,j]P,l][i-l+1/N,j]M$ must end with:

$$\dfrac{\dfrac{\Delta,\Pi\vdash N:B \quad \Gamma',\Gamma'',B,\Pi\vdash P:C}{\Gamma',\Gamma'',\Delta,\Pi\vdash [i-k/N,j]P:C}\,Sub \quad \dfrac{\Delta,\Pi\vdash N:B \quad \Gamma,C,\Gamma'',B,\Pi\vdash M:A}{\Gamma,C,\Gamma'',\Delta,\Pi\vdash [i-l+1/N,j]M:A}\,Sub}{\Gamma,\Gamma',\Gamma'',\Delta,\Pi\vdash [k/[i-k/N,j]P,l][i-l+1/N,j]M:A}\,Sub$$

and its translation is the proof net

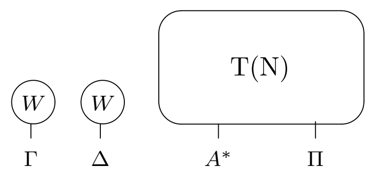


Starting from the first proof net, we eliminate the $c-b$ cut, then the $b-b$ cut, and we apply the equivalence rules A and B to obtain the desired proof net.

— **Rule** d **:** $\langle i\rangle\langle j\rangle M \longrightarrow \langle i+j\rangle M$
The typing environment can be split in three parts Γ, Δ, Π, where i is the length of Γ and j is the length of Δ. The typing judgment of $\langle i\rangle\langle j\rangle M$ ends with

$$\cfrac{\cfrac{\Pi \vdash M : A}{\Delta, \Pi \vdash \langle j\rangle M : A}\,Weak}{\Gamma, \Delta, \Pi \vdash \langle i\rangle\langle j\rangle M : A}\,Weak$$

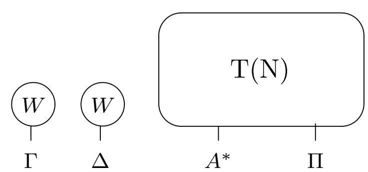
and its translation is the proof net



The typing judgment of $\langle i+j\rangle M$ must end with:

$$\frac{\Pi \vdash M : A}{\Gamma, \Delta, \Pi \vdash \langle i+j\rangle M : A}\,Weak$$

and its translation is the proof net



We notice that the two proof nets are already the same. This is the second of the exception cases of the lemma.

□

4.4. *The proof of strong normalization of* λ_{ws}

We are now able to show strong normalization of λ_{ws}. To achieve this result, we use the following abstract theorem (see for example (Ferreira, Kesner and Puel 1999)) :
Theorem 4.3 Let $R = \langle \mathcal{O}, R_1 \cup R_2\rangle$ be an abstract reduction system such that R_2 is strongly normalizing and there exist a reduction system $S = \langle \mathcal{O}', R'\rangle$, with a translation T of $\mathcal{O}$ into $\mathcal{O}'$ such that $a \longrightarrow_{R_1} b$ implies $T(a) {\longrightarrow^+}_{R'} T(b)$; $a \longrightarrow_{R_2} b$ implies $T(a) = T(b)$. Then if R' is strongly normalizing, $R_1 \cup R_2$ is also strongly normalizing.

If we take $\mathcal{O}$ as the set of typed λ_{ws}-terms, R_1 as $\lambda_{ws} - \{e_2, d\}$, R_2 as $\{e_2, d\}$, $\mathcal{O}'$ as the set of proof nets, T the translation given in Section 4.2 and R' as the reduction R_E, then, by the Theorem 4.3 and the fact that the system including the rules $\{e_2, d\}$ is strongly normalizing (David and Guillaume 1999; David and Guillaume 2001), we can conclude :

Theorem 4.4 (Strong normalization of λ_{ws}) The typed λ_{ws}-calculus is strongly normalizing.

5. A named version of the λ_{ws}-calculus

In this section we present a version of typed λ_{ws} with named variables *à la Church*[§]. We first introduce the grammar of terms, then the typing and reduction rules, and finally, we will briefly discuss the translation of this syntax to PN.

The terms of this calculus are given by the following grammar, where A denotes a type and Γ and Δ denote sets of variables:

$$
\begin{array}{lll}
M ::= & x & \textit{variable} \\
\quad\mid & \lambda x : A.M & \textit{abstraction} \\
\quad\mid & (MM) & \textit{application} \\
\quad\mid & \Delta M & \textit{label} \\
\quad\mid & M[x, M, \Gamma, \Delta] & \textit{substitution}
\end{array}
$$

Intuitively, the term ΔM means that the variables in Δ are not in M, and the term $M[x, N, \Gamma, \Delta]$ means that the variables in Γ do not appear in N (Γ is a subset of the type environment of M, not containing x) and the variables Δ do not appear in M (Δ is a subset of the type environment of N).

Variables are bound by the abstraction and substitution operators, so that for example x is bound in $\lambda x : A.x$ and in $x[x, N, \Gamma, \Delta]$.

Terms are identified modulo α-conversion so that bound variables can be systematically renamed. Indeed, we have $\lambda y : A.y[x, z, \emptyset, \emptyset] =_\alpha \lambda y' : A.y'[x, z, \emptyset, \emptyset]$ and $\lambda y : A.y[x, z, \emptyset, \emptyset] =_\alpha \lambda y : A.y[x', z, \emptyset, \emptyset]$ and $\lambda l : A.y[x, z, \{l\}, \emptyset] =_\alpha \lambda l' : A.y[x, z, \{l'\}, \emptyset]$.

The reduction rules of the calculus with names are given in Figure 4 (notice that rule b_1 is a particular case of rule b_2 with $\Delta = \emptyset$). Remark that these rules may be applied to any term generated by the grammar, and they do not make use of any type information, which is only present in the terms due to their presentation *à la Church*.

The rule f should not be seen as a conditional reduction rule: as we work modulo α-conversion, we can always find a term α-equivalent to an abstraction $\lambda y : A.M$ such that the condition imposed to the rule is true and thus no variable capture arises.

We remark that the conditions on indices used in the typing rules given in Section 4.1 are now conditions on sets of variables. The typing rules are given in Figure 5. Remark that typing environments are managed here as sets (the relative order of variables in the environments does not matter). To make the proofs more readable, we will make a slight abuse of notation by not distinguishing explicitly between type environments (the capital greek letters on the left hand sides of the entailment relation) and sets of variables without type annotations (appearing in the labels of terms and in the explicit substitution constructors).

As we work modulo α-conversion, we can suppose that in the rule $Weak$ the set Δ does not contain variables that are bound in M. We remark that whenever $\Gamma \vdash M[x, N, \Delta, \Pi]$ is derivable, then Γ necessarily contains Δ and Π, which are two *disjoint* sets of variables.

[§] It is of course possible to give a presentation *à la Curry* without type annotations on the abstracted variables.

$$
\begin{array}{llcll}
(b_1) & (\lambda x : A.M)N & \longrightarrow & M[x, N, \emptyset, \emptyset] & \\
(b_2) & (\Delta(\lambda x : A.M))N & \longrightarrow & M[x, N, \emptyset, \Delta] & \\
(f) & (\lambda y : A.M)[x, N, \Gamma, \Delta] & \longrightarrow & \lambda y : A.M[x, N, \Gamma \cup y, \Delta] & \text{if } y \notin FV(N) \\
(a) & (MP)[x, N, \Gamma, \Delta] & \longrightarrow & (M[x, N, \Gamma, \Delta]P[x, N, \Gamma, \Delta]) & \\
(e_1) & \Lambda M[x, N, \Gamma, \Delta] & \longrightarrow & (\Delta \cup (\Lambda \setminus x))M & x \in \Lambda \\
(e_2) & \Lambda M[x, N, \Gamma, \Delta] & \longrightarrow & (\Gamma \cap \Lambda)M[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)] & x \notin \Lambda \\
(n_1) & y[x, N, \Gamma, \Delta] & \longrightarrow & y & y \neq x \\
(n_2) & x[x, N, \Gamma, \Delta] & \longrightarrow & \Gamma N & \\
(c_1) & M[y, P, \Lambda, \Phi][x, N, \Gamma, \Delta] & \longrightarrow & & x \in \Phi \\
 & \multicolumn{3}{c}{M[y, P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)], \Lambda \cap \Gamma, \Delta \cup (\Phi \setminus x)]} & \\
(c_2) & M[y, P, \Lambda, \Phi][x, N, \Gamma, \Delta] & \longrightarrow & M[x, N, (\Gamma \setminus \Phi) \cup y, \Delta \cup (\Phi \setminus \Gamma)] & \\
 & & & \quad [y, P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)], \Lambda \cap \Gamma, \Phi \cap \Gamma] & x \notin \Phi \cup \Lambda \\
(d) & \Gamma\Delta M & \longrightarrow & (\Gamma \cup \Delta)M &
\end{array}
$$

Fig. 4. Reduction Rules of the λ_{ws}-calculus with named variables

$$
\frac{}{\Gamma, x : A \vdash x : A}\ Ax \qquad\qquad \frac{\Gamma \vdash M : A \quad \Gamma \cap \Delta = \emptyset}{\Gamma, \Delta \vdash \Delta M : A}\ Weak
$$

$$
\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash (MN) : A}\ App \qquad\qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B}\ Lamb
$$

$$
\frac{\Delta, \Pi \vdash N : A \quad \Gamma, x : A, \Pi \vdash M : B \quad (\Gamma, x : A) \cap \Delta = \emptyset}{\Delta, \Gamma, \Pi \vdash M[x, N, \Gamma, \Delta] : B}\ Sub
$$

Fig. 5. Typing rules for the λ_{ws}-calculus with named variables

As expected the λ_{ws}-calculus with names enjoys the subject reduction property.

Theorem 5.1 (Subject Reduction) If $\Psi \vdash R : C$ and $R \longrightarrow R'$, then $\Psi \vdash R' : C$.

Proof. The proof proceeds by induction on the structure of the term R.
If the reduction takes place at an internal position of R, it is easy to see that one gets the expected result by applying the induction hypothesis to the reduced subterm.

Otherwise, the reduction takes place at the root of the term R, and we must consider all the possible cases. The pattern of the proof is quite simple: from the shape of R and the fact that $\Psi \vdash R : C$, we determine the last rules applied in the typing derivation, and isolate some subderivations $\pi_1, \ldots \pi_n$ from which it is easy to reconstruct a typing derivation of $\Psi \vdash R' : C$.

— Rule b_1: $R = (\lambda x : A.M)N$ reduces to $M[x, N, \emptyset, \emptyset] = R'$.
  Since $R = (\lambda x : A.M)N$ the typing derivation must necessarily be of the form

$$
\dfrac{\dfrac{\dfrac{\pi_1}{\Psi, x : A \vdash M : C}}{\Psi \vdash \lambda x : A.M : A \to C}\ (Lamb) \qquad \dfrac{\pi_2}{\Psi \vdash N : A}}{\Psi \vdash (\lambda x : A.M)N : C}\ (App)
$$

Now, we can easily build a valid typing derivation for $M[x, N, \emptyset, \emptyset] = R'$ as follows:

$$\dfrac{\dfrac{\pi_1}{\Psi, x : A \vdash M : C} \quad \dfrac{\pi_2}{\Psi \vdash N : A}}{\Psi \vdash M[x, N, \emptyset, \emptyset] : C}\ (Sub)$$

— Rule b_2: $R = (\Delta(\lambda x : A.M))N$ reduces to $M[x, N, \emptyset, \Delta] = R'$. Now, due to the shape of R, the typing derivation must necessarily be of the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma, x : A \vdash M : C}}{\Gamma \vdash \lambda x : A.M : A \to C}\ (Lamb) \quad \Gamma \cap \Delta = \emptyset}{\Gamma, \Delta \vdash \Delta(\lambda x : A.M) : A \to C}\ (Weak) \quad \dfrac{\pi_2}{\Gamma, \Delta \vdash N : A}}{\Gamma, \Delta \vdash (\Delta(\lambda x : A.M))N : C}\ (App)$$

where Ψ is actually split into Γ and Δ. Since x is bound in $\lambda x : A.M$ we can suppose that Δ does not contain x, so that we can construct the derivation

$$\dfrac{\dfrac{\pi_1}{\Gamma, x : A \vdash M : C} \quad \dfrac{\pi_2}{\Gamma, \Delta \vdash N : A} \quad x : A \notin \Delta}{\Gamma, \Delta \vdash M[x, N, \emptyset, \Delta] : C}\ (Sub)$$

— Rule f: $R = (\lambda y : A.M)[x, N, \Gamma, \Delta]$ reduces to $M[x, N, (\Gamma, x : B), \Delta] = R'$, where $y \notin FV(N)$. Due to the shape of R, the typing derivation must necessarily be of the form

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\dfrac{\pi_2}{\Gamma, \Pi, x : B, y : A \vdash M : C}}{\Gamma, \Pi, x : B \vdash \lambda y : A.M : A \to C}\ (Lamb) \quad (\Gamma, x : B) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash (\lambda y : A.M)[x, N, \Gamma, \Delta] : A \to C}\ (Sub)$$

where Ψ is actually split into Γ, Δ and Π. Since y is bound in $\lambda y : A.M$ we can suppose that Δ does not contain y, so that we can construct the derivation

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\pi_2}{\Gamma, \Pi, x : B, y : A \vdash M : C} \quad (\Gamma, x : B, y : A) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi, y : A \vdash M[x, N, (\Gamma, y : A), \Delta] : C}\ (Sub)}{\Gamma, \Delta, \Pi \vdash \lambda y : A.M[x, N, (\Gamma, y : A), \Delta] : A \to C}\ (Lamb)$$

— Rule a: $R = (MP)[x, N, \Gamma, \Delta]$ rewrites to $(M[x, N, \Gamma, \Delta]P[x, N, \Gamma, \Delta]) = R'$ and the typing derivation for R has the shape

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\dfrac{\pi_2}{\Gamma, \Pi, x : B \vdash M : A \to C} \quad \dfrac{\pi_3}{\Gamma, \Pi, x : B \vdash P : A}}{\Gamma, \Pi, x : B \vdash (MP) : C}\ (App) \quad (\Gamma, x : B) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash (MP)[x, N, \Gamma, \Delta] : C}\ (Sub)$$

where Ψ is decomposed into Γ, Δ and Π. Now we can easily construct a derivation π'

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\pi_2}{\Gamma, \Pi, x : B \vdash M : A \to C} \quad (\Gamma, x : B) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash M[x, N, \Gamma, \Delta] : A \to C}\ (Sub)$$

and a derivation π''

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\pi_3}{\Gamma, \Pi, x : B \vdash P : A} \quad (\Gamma, x : B) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash P[x, N, \Gamma, \Delta] : A}\ (Sub)$$

from which we obtain finally

$$\dfrac{\dfrac{\pi'}{\Gamma,\Delta,\Pi \vdash M[x,N,\Gamma,\Delta] : A \to C} \quad \dfrac{\pi''}{\Gamma,\Delta,\Pi \vdash P[x,N,\Gamma,\Delta] : A}}{\Gamma,\Delta,\Pi \vdash (M[x,N,\Gamma,\Delta]P[x,N,\Gamma,\Delta]) : C}\ (App)$$

— Rule e_1: $R = \Lambda M[x,N,\Gamma,\Delta]$ rewrites to $(\Delta \cup (\Lambda \setminus x))M = R'$, where $x \in \Lambda$. Due to the structure of R, the derivation necessarily has the form

$$\dfrac{\dfrac{\pi_1}{\Delta,\Pi \vdash N : B} \quad \dfrac{\dfrac{\pi_2}{\Gamma',\Pi' \vdash M : C} \quad \Lambda \cap (\Gamma',\Pi') = \emptyset}{\Gamma,\Pi,x:B \vdash \Lambda M : C}\ (Weak) \quad (\Gamma,x:B)\cap\Delta = \emptyset}{\Gamma,\Delta,\Pi \vdash \Lambda M[x,N,\Gamma,\Delta] : C}\ (Sub)$$

where Ψ decomposes into Γ,Δ and Π. We know also, by the definition of rule $e1$, that $x \in \Lambda$, so that Λ is actually composed of x plus some other variables coming in part from Γ and in part from Π, that is, $\Lambda = (x : B, \Gamma'', \Pi'')$ whith $\Gamma = \Gamma', \Gamma''$, $\Pi = \Pi', \Pi''$ and such that the set difference $\Gamma \setminus \Lambda$ is Γ' and $\Pi \setminus \Lambda$ is Π'.
Since $\Pi' \subseteq \Pi$, then it is evident that $\Delta \cap \Pi' = \emptyset$, and since $\Gamma' \subseteq \Gamma$, then $\Delta \cap \Gamma' = \emptyset$ comes from the fact that $(\Gamma, x : B) \cap \Delta = \emptyset$. Indeed, $(\Lambda \setminus x) \cap (\Gamma', \Pi') = \emptyset$ is a consequence of the constraint $\Lambda \cap (\Gamma', \Pi') = \emptyset$ in the above typing derivation. We thus obtain

$$\dfrac{\dfrac{\pi_1}{\Gamma',\Pi' \vdash M : C} \quad (\Delta \cup (\Lambda \setminus x)) \cap (\Gamma',\Pi') = \emptyset}{\Gamma,\Delta,\Pi \vdash (\Delta \cup (\Lambda \setminus x))M : C}\ (Weak)$$

— Rule e_2: $R = \Lambda M[x,N,\Gamma,\Delta]$ rewrites to $(\Gamma \cap \Lambda)M[x,N,\Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)] = R'$, where $x \notin \Lambda$.
Due to the structure of R, the typing derivation has necessarily the form

$$\dfrac{\dfrac{\pi_1}{\Delta,\Pi \vdash N : B} \quad \dfrac{\dfrac{\pi_2}{\Gamma',\Pi',x:B \vdash M : C} \quad (\Gamma',\Pi',x:B) \cap \Lambda = \emptyset}{\Gamma,\Pi,x:B \vdash \Lambda M : C}\ (Weak) \quad (\Gamma,x:B)\cap\Delta = \emptyset}{\Gamma,\Delta,\Pi \vdash \Lambda M[x,N,\Gamma,\Delta] : C}\ (Sub)$$

where Ψ decomposes into Γ, Δ and Π. Furthermore, by the definition of rule e_2, we have that $x \notin \Lambda$, so that Λ can be written as Γ'', Π'', where $\Gamma = \Gamma', \Gamma''$, $\Pi = \Pi', \Pi''$, and this means that $\Gamma' = \Gamma \setminus \Lambda$, $\Pi'' = \Lambda \setminus \Gamma$ and $\Gamma', \Pi', \Lambda = \Gamma, \Pi$ and $(\Gamma \cap \Lambda) \cup \Gamma' = \Gamma'' \cup \Gamma' = \Gamma$.
We can then build the required derivation

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Delta,\Pi \vdash N : B} \quad \dfrac{\pi_2}{\Gamma',\Pi',x:B \vdash M : C}}{\Delta,\Pi,\Gamma' \vdash M[x,N,\Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)] : C}\ (Sub)}{\Gamma,\Delta,\Pi \vdash (\Gamma \cap \Lambda)M[x,N,\Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)] : C}\ (Weak)$$

Notice that one has to check the side conditions of the typing rules. For $(Weak)$, we need $(\Gamma \cap \Lambda) \cap (\Delta, \Pi, \Gamma') = \emptyset$, but $(\Gamma \cap \Lambda) = \Gamma''$, and $\Gamma'' \cap (\Delta, \Pi, \Gamma') = \emptyset$ because $\Gamma = \Gamma', \Gamma''$ and (Γ, Δ, Π) are well-formed environments. For (Sub), we need $(\Gamma', x : B) \cap (\Delta, \Pi'') = \emptyset$. First of all we notice that $(\Gamma, x : B) \cap \Delta = \emptyset$ holds because of the side condition of the (Sub) rule in the typing derivation for R. Secondly, $\Gamma' \cap \Pi'' = \emptyset$ holds because (Γ, Δ, Π) is a well formed environment. Last, $x : B \notin \Pi''$ holds because $x : B \notin \Lambda$ (definition of rule e_2) and $\Pi'' = \Lambda \setminus \Gamma$.

— Rule n_1: $R = y[x,N,\Gamma,\Delta]$ rewrites to $y = R'$. From the structure of R, we know that the derivation must be of the form

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{}{\Gamma, \Pi, x : B \vdash y : C}\ (Ax) \quad (\Gamma, x : B) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash y[x, N, \Gamma, \Delta] : C}\ (Sub)$$

where Ψ decomposes into Γ, Δ and Π. The seeked derivation is then simply

$$\dfrac{}{\Gamma, \Delta, \Pi \vdash y : C}\ (Ax)$$

— Rule n_2: $R = y[x, N, \Gamma, \Delta]$ rewrites to $\Gamma N = R'$

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : C} \quad \dfrac{}{\Gamma, \Pi, x : C \vdash x : C}\ (Ax) \quad (\Gamma, x : C) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash y[x, N, \Gamma, \Delta] : C}\ (Sub)$$

where Ψ decomposes into Γ, Δ and Π. Now, the side condition for (Sub) tells us that $\Gamma \cap \Delta = \emptyset$, and Γ, Π is well formed, so we can conclude that $\Gamma \cap (\Delta, \Pi) = \emptyset$, so we can build the required derivation as follows

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : C} \quad \Gamma \cap (\Delta, \Pi) = \emptyset}{\Gamma, \Delta, \Pi \vdash \Gamma N : C}\ (Weak)$$

— Rule c_1: $R = M[y, P, \Lambda, \Phi][x, N, \Gamma, \Delta]$ rewrites to $M[y, P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)], \Lambda \cap \Gamma, \Delta \cup (\Phi \setminus x)] = R'$, where $x \in \Phi$.
From the structure of R, we know that the derivation must be of the form

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\pi'}{\Gamma, x : B, \Pi \vdash M[y, P, \Lambda, \Phi] : C} \quad (\Gamma, x : B) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash M[y, P, \Lambda, \Phi][x, N, \Gamma, \Delta] : C}\ (Sub)$$

where Ψ decomposes into Γ, Δ and Π. Now, π' is a derivation that necessarily ends with an application of the (Sub) rule, so that the environment $\Gamma, x : B, \Pi$ gets split into several subparts that are used to type the terms M and P. Looking at the shape of the (Sub) rule, we see that in general this splitting divides Π into three pairwise disjoint components (each of which possibly empty): Π_Λ, that is part of Λ, Π_Φ, which is part of Φ, and a Π' which is common to the typing environments used to type M and P. Similarly, Γ decomposes into pairwise disjoint Γ_Λ, Γ_Φ and Γ'. And then $\Lambda = \Gamma_\Lambda \cup \Pi_\Lambda$ and $\Phi = \Gamma_\Phi \cup \Pi_\Phi \cup x : B$. We also know that x is in the typing environment of P since $x \in \Phi$, so it must appear in the typing environment of P.
To sum all this up, the derivation π' must be of the form

$$\dfrac{\dfrac{\pi_2}{\Gamma', \Gamma_\Phi, x : B, \Pi', \Pi_\Phi \vdash P : A} \quad \dfrac{\pi_3}{\Gamma', \Gamma_\Lambda, y : A, \Pi', \Pi_\Lambda \vdash M : C} \quad (\Lambda, y : A) \cap \Phi = \emptyset}{\Gamma, x : B, \Pi \vdash M[y, P, \Lambda, \Phi] : C}\ (Sub)$$

Now, from π_1 and π_2 we can first of all build the derivation π'', where we use the fact that, since Π and Γ are disjoint, and $\Lambda = \Gamma_\Lambda \cup \Pi_\Lambda$, we know that Π_Λ can be written as $\Lambda \setminus \Gamma$

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\pi_2}{\Gamma', \Gamma_\Phi, x : B, \Pi', \Pi_\Phi \vdash P : A} \quad (\Gamma', \Gamma_\Phi, x : B) \cap (\Delta \cup (\Lambda \setminus \Gamma)) = \emptyset}{\Delta, \Pi_\Lambda, \Gamma', \Gamma_\Phi, \Pi', \Pi_\Phi \vdash P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)] : A}\ (Sub)$$

Notice that the side condition holds because, on one side, Π and Γ are disjoint, on the other side $x \notin \Pi$ since $\Gamma, x : B, \Pi$ is well-formed, and finally because we know from the side conditions of the typing derivation for R that Δ is disjoint from $\Gamma, x : B$.

Now, since y is bound in $M[y, P, \Lambda, \Phi]$, we can also suppose that Δ does not contain y, so that we can build the following derivation π'''

$$\dfrac{\dfrac{\pi''}{\Delta, \Pi_\Lambda, \Gamma', \Gamma_\Phi, \Pi', \Pi_\Phi \vdash P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)] : A} \quad \dfrac{\pi_3}{\Gamma', \Gamma_\Lambda, y : A, \Pi', \Pi_\Lambda \vdash M : C}}{\Delta, \Gamma', \Gamma_\Lambda, \Gamma_\Phi, \Pi', \Pi_\Lambda, \Pi_\Phi \vdash M[y, P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)], \Gamma_\Lambda, \Delta \cup (\Gamma_\Phi \cup \Pi_\Phi)] : C} \ (Sub)$$

Where the side condition reads

$$(y : A, \Gamma_\Lambda) \cap (\Delta, \Gamma_\Phi, \Pi_\Phi) = \emptyset$$

This holds because, on one side, Γ_Λ is disjoint from Γ_Φ by definition, disjoint from Π_Φ because $\Gamma \cap \Pi = \emptyset$ and disjoint from Δ because $\Gamma \cap \Delta = \emptyset$ by the side conditions of the typing derivation for R; on the other side, we have already seen that we can assume y is not in Δ and we know from the side conditions in the typing derivation of R that y is not in Φ, which is precisely $\Gamma_\Phi \cup \Pi_\Phi$.

Now, the conclusion of this derivation, once we apply all the equalities of environments that we have established up to now, actually becomes

$$\Delta, \Gamma, \Pi \vdash M[y, P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)], \Lambda \cap \Gamma, \Delta \cup (\Phi \setminus x)] : C$$

so that π''' is the seeked derivation.

— Rule c_2: the term $R = M[y, P, \Lambda, \Phi][x, N, \Gamma, \Delta]$ rewrites to the term $M[x, N, (\Gamma \setminus \Phi) \cup y, \Delta \cup (\Lambda \setminus \Gamma)][y, P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Phi \setminus \Gamma)], \Lambda \cap \Gamma, \Gamma \cap \Phi] = R'$, where $x \notin \Phi \cup \Lambda$.
From the structure of R, we know that the derivation must be of the form

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\pi'}{\Gamma, x : B, \Pi \vdash M[y, P, \Lambda, \Phi] : C} \quad (\Gamma, x : B) \cap \Delta = \emptyset}{\Gamma, \Delta, \Pi \vdash M[y, P, \Lambda, \Phi][x, N, \Gamma, \Delta] : C} \ (Sub)$$

where Ψ decomposes as Γ, Δ, Π. As for the rule c_1 the environment Γ decomposes as $\Gamma_\Lambda, \Gamma_\Phi, \Gamma'$; the environment Π decomposes as $\Pi_\Lambda, \Pi_\Phi, \Pi'$; $\Lambda = \Gamma_\Lambda \cup \Pi_\Lambda$ and $\Phi = \Gamma_\Phi \cup \Pi_\Phi$. We also know that $x \notin \Phi \cup \Lambda$, so it must appear in the common typing environment of P and M.
The derivation π' must be of the form

$$\dfrac{\dfrac{\pi_2}{\Gamma', \Gamma_\Phi, x : B, \Pi', \Pi_\Phi \vdash P : A} \quad \dfrac{\pi_3}{\Gamma', \Gamma_\Lambda, y : A, x : B, \Pi', \Pi_\Lambda \vdash M : C} \quad (\Lambda, y : A) \cap \Phi = \emptyset}{\Gamma, x : B, \Pi \vdash M[y, P, \Lambda, \Phi] : C} \ (Sub)$$

We know that Π_Λ can be written as $\Lambda \setminus \Gamma$ and Π_Φ can be written as $\Phi \setminus \Gamma$. Now, we can first of all build the derivation π''

$$\dfrac{\dfrac{\pi_1}{\Delta, \Pi \vdash N : B} \quad \dfrac{\pi_2}{\Gamma', \Gamma_\Phi, x : B, \Pi', \Pi_\Phi \vdash P : A} \quad (\Gamma', \Gamma_\Phi, x : B) \cap (\Delta \cup (\Lambda \setminus \Gamma)) = \emptyset}{\Delta, \Pi_\Lambda, \Gamma', \Gamma_\Phi, \Pi', \Pi_\Phi \vdash P[x, N, \Gamma \setminus \Lambda, \Delta \cup (\Lambda \setminus \Gamma)] : C} \ (Sub)$$

As in c_1, the side condition holds.
Since y is bound in $M[y, P, \Lambda, \Phi]$, then we can suppose by α-conversion that Δ does not contain y, and we also obtain the derivation π'''

$$\dfrac{\dfrac{\pi_1}{\Delta,\Pi\vdash N:B} \quad \dfrac{\pi_3}{\Gamma',\Gamma_\Lambda,y:A,x:B,\Pi',\Pi_\Lambda\vdash M:C} \quad ((\Gamma\setminus\Phi)\cup y:A\cup x:B)\cap(\Delta\cup(\Phi\setminus\Gamma))=\emptyset}{\Gamma',\Gamma_\Lambda,y:A,\Delta,\Pi\vdash M[x,N,(\Gamma\setminus\Phi)\cup y,\Delta\cup(\Phi\setminus\Gamma)]:C}\ (Sub)$$

Also here the side conditions holds: on one hand Γ and Δ are disjoint and $(\Gamma\setminus\Phi)$ and $(\Phi\setminus\Gamma)$ are trivially disjoint, on the other hand y is not in Δ, by α-conversion and x is not in Δ by hypothesis on the side condition of the type derivation of R, finally y is not in Φ by the side condition of derivation π' and x is not in Φ by hypothesis of the rule c_2. Now, since $(\Lambda, y:A)\cap\Phi=\emptyset$, by the side condition of the (Sub) rule in the derivation π', we can finally build

$$\dfrac{\dfrac{\pi''}{\Gamma',\Gamma_\Phi,\Delta,\Pi\vdash P[x,N,\Gamma\setminus\Lambda,\Delta\cup(\Lambda\setminus\Gamma)]:A} \quad \dfrac{\pi'''}{\Gamma',\Gamma_\Lambda,y:A,\Delta,\Pi\vdash M[x,N,(\Gamma\setminus\Phi)\cup y,\Delta\cup(\Phi\setminus\Gamma)]:C}}{\Gamma,\Delta,\Pi\vdash M[x,N,(\Gamma\setminus\Phi)\cup y,\Delta\cup(\Phi\setminus\Gamma)][y,P[x,N,\Gamma\setminus\Lambda,\Delta\cup(\Lambda\setminus\Gamma)],\Lambda\cap\Gamma,\Phi\cap\Gamma]:C}\ (Sub)$$

which is the required derivation.

— Rule d: $R=\Gamma\Delta M$ rewrites to $(\Gamma\cup\Delta)M=R'$ The derivation of R must be of the form

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Pi\vdash M:C} \quad \Delta\cap\Pi=\emptyset}{\Delta,\Pi\vdash\Delta M:C}\ (Weak) \quad \Gamma\cap(\Delta,\Pi)=\emptyset}{\Gamma,\Delta,\Pi\vdash\Gamma\Delta M:C}\ (Weak)$$

with Ψ that decomposes into Γ, Δ and Π.
It is easy to rebuild the required derivation

$$\dfrac{\dfrac{\pi_1}{\Pi\vdash M:C} \quad (\Gamma\cup\Delta)\cap\Pi=\emptyset}{\Gamma,\Delta,\Pi\vdash(\Gamma\cup\Delta)M:C}\ (Weak)$$

□

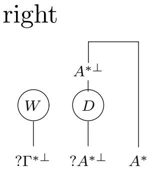
6. Strong normalization of the λ_{ws} calculus with names

We now give the translation of the terms of λ_{ws} with names into proof nets in PN, and the proof of strong normalization of λ_{ws}.

In order to translate a term of λ_{ws} into a proof net, we use exactly the same translation of types that we used in Section 4.2 and we then define the translation of a term M using the type derivation of M.

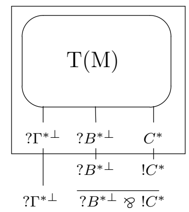
— If the term is a variable and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\dfrac{}{\Gamma,x:A\vdash x:A}\ Ax$$



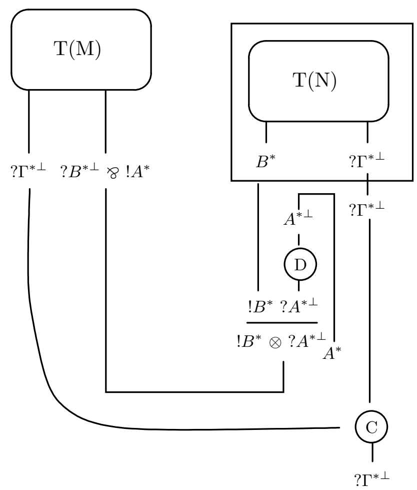
— If the term is a λ-abstraction and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\frac{\Gamma, x : B \vdash M : C}{\Gamma \vdash \lambda x : B.M : B \to C}\ Lamb$$



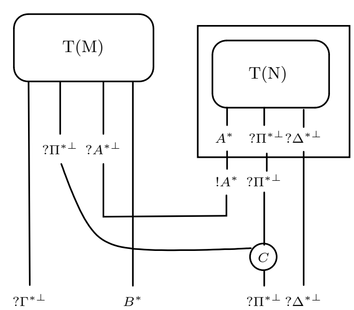
— If the term is an application and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash (MN) : A}\ App$$



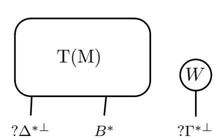
— If the term is a substitution and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\frac{\Delta, \Pi \vdash N : A \quad \Gamma, x : A, \Pi \vdash M : B}{\Delta, \Gamma, \Pi \vdash M[x, N, \Gamma, \Delta] : B}\ Sub$$



— Finally, if the term is a label and its typing judgement ends with the rule written below on the left, then its translation is the proof net on the right

$$\frac{\Delta \vdash M : B}{\Gamma, \Delta \vdash \Gamma M : B}\ Weak$$



We can clearly verify that the translation is identical to that given for λ_{ws} with de Bruijn indices. This is not surprising since the type derivations are similar in both formalisms.

The simulation of the reduction rules of the λ_{ws}-calculus with names by the reduction R_E is identical to that given in Section 4.2 for the λ_{ws}-calculus with indices. We just remark that rule n_3 has no sense in the formalism with names so that the proof has one less case. We just state the result without repeating a boring verification:

Lemma 6.1 (Simulation of λ_{ws} with names) If t λ_{ws}-reduces to t' in the formalism with names, then $T(t) \longrightarrow^+_{R_E} T(t')$, except for the rules e_2 and d for which we have $T(t) = T(t')$.

We can then conclude the following:

Theorem 6.2 (Strong Normalization of λ_{ws} with names) The typed λ_{ws}-calculus with names is strongly normalizing.

7. Conclusion and future works

In this paper we enriched the standard notion of cut elimination in proof nets in order to obtain a system R_E which is flexible enough to provide an interpretation of λ-calculi with explicit substitutions and which is much simpler than the one proposed in (Di Cosmo and Kesner 1997). We have proved that this system is strongly normalizing.

We have then proposed a natural translation from λ_{ws} into proof nets that immediately provides strong normalization of the typed version of λ_{ws}, a calculus featuring full composition of substitutions. The proof is extremely simple w.r.t the proof of PSN of λ_{ws} given in (David and Guillaume 1999; David and Guillaume 2001) and shows in some sense that λ_{ws}, which was designed independently of proof nets, is really tightly related to reduction in proof nets.

Finally, the fact that the relative order of variables is lost in the proof-net representation of a term lead us to discover a version of typed λ_{ws} with named variables, instead of de Bruijn indices. This typed named version of λ_{ws} gives a better understanding of the mechanisms of the calculus. In particular, names allow to understand the manipulation of explicit weakenings in λ_{ws} without entering into the details of renaming of de Bruijn indices. However, the study of the properties of reduction, such as confluence and PSN, for non-typed or non well-formed terms with names remains as further work.

This work suggests several interesting directions for future investigation: on the linear logic side, one should wonder whether R_E is the definitive system able to interpret β reduction, or whether we need some more equivalences to be added. Indeed, there are still a few cases in which the details of a sequent calculus derivation are inessential, even if we did not need to consider them for the purpose of our work, like for example

$$\dfrac{\dfrac{\vdash \Gamma, B}{\vdash ?A, \Gamma, B}\ Weakening}{\vdash ?A, \Gamma, !B}\ Box \qquad\qquad \dfrac{\dfrac{\vdash \Gamma, B}{\vdash \Gamma, !B}\ Box}{\vdash ?A, \Gamma, !B}\ Weakening$$

On the explicit substitutions side, we look forward to the discovery of a calculus with multiple substitutions with the same properties as λ_{ws}, in the spirit of λ_σ.

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