Provable isomorphisms of types

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Abstract

A constructive characterization is given of the isomorphisms which must hold in all models of the typed lambda calculus with surjective pairing. By the close relation between

Definition 1.1 $Th^1_{\times T}$ is a theory of equality plus the following axiom schemas, where **T** is a constant symbol:

1. $A \times B = B \times A$
2. $A \times (B \times C) = (A \times B) \times C$
3. $(A \times B) \to C = A \to (B \to C)$
4. $A \to (B \times C) = (A \to B) \times (A \to C)$
5. $A \times \mathbf{T} = \mathrm{A}$
6. $A \to \mathbf{T} = \mathbf{T}$
7. $\mathbf{T} \to A = \mathrm{A}$

The Main Theorem of this paper shows that two types A and B can be constructively proved to be isomorphic, by two programs which act one as the inverse of the other, iff $Th^1_{\times T} \vdash \mathrm{A} = \mathrm{B}$.

In order to discuss the soundness of Th, and explain where it comes from, we hint here of its categorical meaning. Note, though, that no notion nor result from Category Theory is used in most of the paper.

Since models of the typed lambda calculus with surjective pairing are exactly the Cartesian closed categories (CCC), our results translate directly into theorems on when two generic objects are isomorphic in all CCC's. In other words, $Th^1_{\times T}$ characterizes which are isomorphic just by the Cartesian closed structure of the category in which they are interpreted, no matter which particular CCC is chosen.

Observe first that $Th^1_{\times T}$ is realized in every Cartesian Closed Category, when "=" is interpreted as isomorphism. The first three axioms describe properties of the Cartesian product (associativity, commutativity, identity for $\times$), and the second three axioms can be seen as the properties of the three adjunctions of a CCC that relate product, exponent and the terminal object. The last equation ($\mathbf{T} \to A = \mathrm{A}$) tells us that the arrows from the terminal object to A in a CCC are the points of A. Thus, the theory $Th^1_{\times T}$ is sound. A consequence of our main result is the completeness of $Th^1_{\times T}$ with respect to CCC's. That is, no other isomorphism is valid in all CCC's. (This is not obvious because there are categorical models of $Th^1_{\times T}$ which are not CCC's: take a Cartesian Category with a bifunctor " $\to$ " such that $A \to B = \mathrm{B}$, say).

A further consequence of the work below in λ-calculus will be an insight into the composition of derivations in Proof Theory. The typed lambda calculus with surjective pairing is the language for proofs of $IPC(\mathbf{True}, \wedge, \to)$, the intuitionistic positive propositional calculus. In the proof theoretic framework we then characterize *equivalent formulae*, where two formulae A and B are considered equivalent if, given a proof f of the sequent $\mathrm{A} \vdash \mathrm{B}$, and a proof g of the sequent $\mathrm{B} \vdash \mathrm{A}$, $g\ f$ yields, after cut-elimination, the identity proof of the sequent $\mathrm{A} \vdash \mathrm{A}$ and vice-versa. The details of both the categorical and proof-theoretic applications are discussed in [DCL91].

As an example of the use of such results in computer science we note the two papers by Rittri ([Rit91], [Rit90]) in which the author discusses the problem of finding applicable functions in a program library. For example, one might be interested in looking up various search functions. As a result it might be useful to inspect all functions which take an element and a table and return an index to the table. Because trivial differences in argument order or Currying may lead one to ignore useful functions, it is important to be able to find all those functions whose type is isomorphic to that for which one is searching.

Rittri's application of the result presented here settles on the same notion of provable isomorphism. He cites the paper by Solv'ev ([Sol83]), in which the author presents the same result as in our main theorem (Theorem 4.9), although by an entirely different proof which is

based on taking the natural numbers as objects in a CCC (with $\times$ interpreted as multiplication and $\rightarrow$ as exponentiation) and then showing the equational completeness of the theory of $(N, 1, \times, \uparrow)$. (Meyer and Statman, *personal communication*, suggested a similar proof for the exponential fragment only; also the abstract in Martin ([Mar72]) states the same fact). Solv'ev also provides a decision procedure similar to that given here.

We note that in a forthcoming paper, the second author extends these results to the second-

alpha-beta-eta-csi:

(α) λx:A.M = λy:A.M[x:=y], if y is free for x in M

($\rightarrow \beta$) (λx:A.M)N = M[x:=N], if N is free for x in M

($\rightarrow \eta$) λx:A.(Mx) = M, if $x \notin FV(M)$

(ξ) if M=N then λx:A.M = λx:A.N

surjective pairing:

($\times\beta_1$) $\mathrm{p}_1(\langle M, N\rangle) = \mathrm{M}$

($\times\beta_2$) $\mathrm{p}_2(\langle M, N\rangle) = \mathrm{N}$

($\times\eta$) $\langle \mathrm{p}_1(M), \mathrm{p}_2(M)\rangle = \mathrm{M}$

terminal object:

($*$) If M : $A \rightarrow \mathbf{T}$ then M = $*_A$.

Notation 2.3 Given a sequence $\mathrm{M}_1, \ldots, \mathrm{M}_n$ of terms, and sequence $\mathrm{x} = \mathrm{x}_1, \ldots, \mathrm{x}_n$ of variables, $\mathrm{N}[\overrightarrow{M}/\overrightarrow{x}]$ denotes the simultaneous substitution of every term M_i for the variable x_i in the term N (for simplicity, we always assume bound variables are renamed as necessary to avoid capture of free variables). We also use the notation $\mathrm{N}[\mathrm{M}/\overrightarrow{x}]$ to express the simultaneous substitution of the term M for all the variables in $\overrightarrow{x}$. For application we follow the usual convention of associating to the left, i.e. $\mathrm{N}_1 \ldots \mathrm{N}_n$ is to be parsed as $(\ldots(N_1 N_2)\ldots N_n)$. In case a substitution is applied only to a subsequence of an application $\mathrm{M}_1 \ldots \mathrm{M}_n$, we will use the notation $N_1 N_2 \ldots \{N_i \ldots N_k[\overrightarrow{M}/\overrightarrow{x}]\}\ldots N_n$ to denote the term $\mathrm{N}_1 \ldots \mathrm{N}_n$ with the substitution $[\overrightarrow{M}/\overrightarrow{x}]$ applied only to the terms $\mathrm{N}_i \ldots \mathrm{N}_k$.

We write $\langle M_1, \ldots M_n\rangle$ for $\langle \ldots \langle\langle M_1, M_2\rangle, M_3\rangle, \ldots\rangle$.

$\lambda^1\beta\eta\pi$ is the calculus without terminal object and related rules, $\lambda^1\beta\eta$ is the classical typed calculus, and $\lambda_{\beta\eta}$ the type-free calculus. Finally, let $I_A = \lambda$x:A.x be the identity of type A.

Remark 2.4 *Notion of reduction for $\lambda^1\beta\eta\pi*$. The notion of reduction associated with the equational theory of $\lambda^1\beta\eta\pi*$ obtained by just orienting the equalities in the axioms to the right is not Church-Rosser. It is possible, though, to derive for this equality theory another notion of reduction that has the Church-Rosser property; in the following we will refer to this latter one when talking about reduction, normal forms, and so on for $\lambda^1\beta\eta\pi*$ (see [Pot81], [CDC91]).*

Definition 2.5 Let $A, B \in \mathbf{Tp}$. Then A and B are **provably isomorphic** ($A \cong_p B$) iff there exist closed λ-terms M : $A \rightarrow B$ and N : $B \rightarrow A$ such that $\lambda^1\beta\eta\pi* \vdash$ M N = I_B and $\lambda^1\beta\eta\pi* \vdash$ N M = I_A. We then say that M and N are **invertible** terms, and that M is an inverse of N, in $\lambda^1\beta\eta\pi*$.

Note that, as usual, the inverse of a term M (if it exists) is unique up to "=." Suppose that types A and B are provably isomorphic and consistently substitute arbitrary types for the common base types. Then the isomorphism still holds: just use the corresponding terms with updated types. Borrowing terminology from Statman (1983) we may say that the notion of provable isomorphism is typically ambiguous.

Theorem 2.6 (Main Theorem (*easy implication*)) $Th^1_{\times T} \vdash A = B \Rightarrow A \cong_p B$.

Proof. We give the terms associated to each axiom and rule. As $Th^1_{\times T}$ is a theory of equality, one has first to observe that the usual axioms and inference rules yield and preserve provable isomorphisms:

- λx:A.x proves A = A;

- if M, with inverse N, proves A = B , then N proves B = A;
- if an invertible M proves A = B and an invertible N proves B = C, then the term N∘M = λx:A.N (M x), that is clearly invertible, proves A = C ;
- if an invertible term M proves A = B and an invertible term N proves C = D, then the invertible term λx:$A \times C.\langle M(\mathrm{p}_1 x), N(\mathrm{p}_2 x)\rangle$ proves $A \times C = B \times D$;
- if an invertible M proves A = B and an invertible N proves C = D, then λy:$A \to C.\lambda$x:B.N (y (M^{-1} x)), where M^{-1} is the inverse of M, proves $A \to C = B \to D$ and it is invertible (take λy:$B \to D.\lambda$x:A.N^{-1} (y (M x))).

We next check the proper axioms:

1. $A \times B = B \times A$ is proved by λx:$A \times B.\langle \mathrm{p}_2 x, \mathrm{p}_1 x\rangle$;
2. $A \times (B \times C) = (A \times B) \times C$ is proved by λx: $A \times (B \times C).\langle\langle \mathrm{p}_1 x, \mathrm{p}_1(\mathrm{p}_2 x)\rangle, \mathrm{p}_2(\mathrm{p}_2 x)\rangle$, that is invertible;
3. $(A \times B) \to C = A \to (B \to C)$ is proved by λz:$(A \times B) \to C.\lambda$x:A.$\lambda$y:B.z$\langle x, y\rangle$ with inverse λz:$A \to (B \to C).\lambda$x:$A \times B$.z (p_1x) (p_2x);
4. $A \to (B \times C) = (A \to B) \times (A \to C)$ is proved by
   λz:$A \to (B \times C).\langle \lambda x : A.(\mathrm{p}_1(zx)), \lambda x : A.(\mathrm{p}_2(zx))\rangle$
   with inverse λz:$(A \to B) \times (A \to C).\lambda$x:A.$\langle(\mathrm{p}_1 z)x, (\mathrm{p}_2 z)x\rangle$;
5. $A \times \mathbf{T} = \mathrm{A}$ is proved by p_1 with inverse λx:A.$\langle x, *_A x\rangle$ (to check invertibility, notice that $*_A \circ \mathrm{p}_1 = *_{A \to \mathbf{T}} = \mathrm{p}_2$);
6. $A \to \mathbf{T} = \mathbf{T}$ is proved by $*_{(A \to \mathbf{T})}$ with inverse λx:$\mathbf{T}.*_A$;
7. $\mathbf{T} \to A = \mathrm{A}$ is proved by λz:$\mathbf{T} \to A.\mathrm{z}(*_{(\mathbf{T} \to A)}\mathrm{z})$ with inverse λx:A.λy:$\mathbf{T}$.x.

□

The rest of this section, as well sections 3 and 4, are dedicated to the proof of the other implication of the Main Theorem. The first steps are done by reducing types to a "type normal form". The axioms of $Th^1_{\times T}$ suggest the following rewrite system **R** for types (essentially $Th^1_{\times T}$ without commutativity):

Definition 2.7 [Type rewriting **R**]
Let "$\rightsquigarrow$" be the transitive and substitutive **type-reduction** relation generated by:

1. $A \times (B \times C) \rightsquigarrow (A \times B) \times C$
2. $(A \times B) \to C \rightsquigarrow A \to (B \to C)$
3. $A \to (B \times C) \rightsquigarrow (A \to B) \times (A \to C)$
4. $A \times \mathbf{T} \rightsquigarrow \mathrm{A}$
5. $\mathbf{T} \times A \rightsquigarrow \mathrm{A}$
6. $A \to \mathbf{T} \rightsquigarrow \mathbf{T}$
7. $\mathbf{T} \to A \rightsquigarrow \mathrm{A}$

The system **R** yields an obvious notion of **normal form for types** (type-n.f.), i.e. when no type reduction is applicable. Note that 4, 5 and 6 "eliminate the **T**'s", while 2 and 3 "bring the $\times$ outside". It is then easy to observe that each type-n.f. is **T** or has the structure $\mathrm{S}_1 \times \ldots \times \mathrm{S}_n$ where each S_i does not contain **T** or "$\times$". We write **nf(S)** for the normal form of S (there is exactly one, see 2.8), and say that a normal form is non-trivial if it is not **T**.

Proposition 2.8 *Each type has a unique type normal form in* **R**.

Proof. Notice that in any **R**-reduction, starting with a given type S:

(i) Rules 2 and 3 can be applied only finitely many times, as they strictly decrease the number of $\times$'s in the scope of an arrow of S and this number is finite and is not increased by any other rule.

(ii) Between an application of rule 2 or 3 (yielding type S') and the next one, the remaining rules can be applied only finitely many times (4, 5, 7 and 6 simply throw away some subformula reducing by one the number of products or arrows, which is finite; rule 1 is just associativity to the left).

So, after a finite reduction path we get a type S" with no redex for rules 2 and 3, and then, again, the remaining rules can be applied only finitely many times (at most the length of S" plus the times required for associating S" to the left). The resulting type nf(S) has then no products in the scope of any arrow (otherwise 2 and 3 could be applied), and is either **T** or a type with no occurrence of **T** (otherwise 4, 5, 7 and 6 could be applied). Thus nf(S) is a product of types, each of which has no occurrence of $\times$.

It is easy to observe that **R** is Church-Rosser too and, thus, that nf(S) is unique. (Note also that we have actually proved that **R** strongly normalizes) □

From the implication proved above of the Main Theorem, since $\mathbf{R} \vdash \mathrm{S} \rightsquigarrow \mathrm{R}$ implies $Th^1_{\times T} \vdash \mathrm{S} = \mathrm{R}$, it is clear that any reduction $\mathbf{R} \vdash \mathrm{S} \rightsquigarrow \mathrm{R}$ is witnessed (or, proved, in the "types-as-propositions" analogy) by an invertible term of type S

3 More Lemmas: From $\lambda^1\beta\eta\pi*$ to the Classical $\lambda^1\beta\eta$

This is a technical section, where we display the statements of some crucial lemmas. Their proofs are postponed to the appendix. Our aim is to reduce invertibility in $\lambda^1\beta\eta\pi*$ to invertibility in $\lambda^1\beta\eta$.

Recall first that, when $Th^1_{\times T} \vdash \mathrm{S} = \mathrm{R}$, one has $nf(S) \equiv \mathbf{T} \equiv nf(R)$, or $Th^1_{\times T} \vdash nf(S) \equiv \mathrm{S}_1 \times \ \ldots \times \mathrm{S}_n = \mathrm{R}_1 \times \ \ldots \times \mathrm{R}_m \equiv nf(R)$. Notice now that, in the latter case, there cannot be any occurrence of $\mathbf{T}$ in either type. Indeed, a non trivial type-n.f. cannot be provably equated to $\mathbf{T}$, as can be easily seen by taking a non-trivial model. Thus we restrict our attention to equations like $\mathrm{S}_1 \times \ \ldots \times \mathrm{S}_n = \mathrm{R}_1 \times \ \ldots \times \mathrm{R}_m$ with no occurrence of $\mathbf{T}$ and, hence, to invertible terms with no occurrence of the type constant $\mathbf{T}$ in their types. We can show that these terms do not contain any occurrence of $*_A$ either, for any type A, via the following lemmas.

Lemma 3.1 (Form of the terms of a product type) *Given a term M of $\lambda^1\beta\eta\pi*$ in normal form such that M: $A \times B$, then either $M \equiv \langle M_1, M_2\rangle$, for some M_1, M_2, or there is a free variable $x : C$ in M such that $A \times B$ is a type subexpression of C.*

Proof. By induction on the length of the structure of M (see appendix). □

Lemma 3.2 (There are no $*_A$ in a term in n.f. if its type does not contain T) *Assume that in a term M of $\lambda^1\beta\eta\pi*$ in normal form there is an occurrence of $*_A$, for some type A. Then there is some occurrence of the type constant $\mathbf{T}$ in the type of M or in the type of some free variable of M.*

Proof. By induction on the structure of M (see appendix). □

Proposition 3.3 (Isomorphisms between type-n.f.'s are given by terms in $\lambda^1\beta\eta\pi$) *Assume that S and R are non trivial type-n.f.'s. If the closed terms M and N prove $S \cong_p R$ in $\lambda^1\beta\eta\pi*$, then their normal forms contain no occurrences of the constants $*_A$. (Thus, M and N are actually in $\lambda^1\beta\eta\pi$).*

Proof. By the previous lemma, as the terms are closed and no $\mathbf{T}$ occurs in their type. □

So we have factored out the first class of constants $*_A$, and we have restricted ourselves to $\lambda^1\beta\eta\pi$. In the next step we eliminate pairing as well, in a sense.

There is a problem though. Our aim is to reduce the investigation of invertible terms in $\lambda^1\beta\eta\pi*$ to that of terms in $\lambda^1\beta\eta\pi$. This is done on the grounds of Proposition 2.10 by examining each component of the product, where the isomorphism will be given by terms of $\lambda^1\beta\eta$. However, in the notation of Proposition 2.10, consider the term M' : $nf(A) \to nf(B)$. M' is invertible in (the equational theory of) $\lambda^1\beta\eta\pi*$ and, thus, also the subterms yielding the isomorphism of the components (see 3.7 and 3.8 below) are, a priori, invertible in $\lambda^1\beta\eta\pi*$, while we need to know that they are actually invertible in $\lambda^1\beta\eta$. We get rid of the problem by the following remark.

Remark 3.4 *(The equational theory of) $\lambda^1\beta\eta\pi*$ is a conservative extension of (the equational theory of) $\lambda^1\beta\eta$. Similarly for $\lambda^1\beta\eta\pi$ with respect to $\lambda^1\beta\eta$.*

Indeed, both $\lambda^1\beta\eta\pi*$ and $\lambda^1\beta\eta\pi$ are Church-Rosser, where "the theory of reduction" for $\lambda^1\beta\eta\pi$ is given by orienting the equalities in the axioms from left to right (for the C-R property see the references in the remark before 2.5) . Consider now M and N in $\lambda^1\beta\eta\pi$ such that $\lambda^1\beta\eta\pi*\vdash \mathrm{N} = \mathrm{M}$ and let P be the common reductum. Then $\lambda^1\beta\eta\pi*\vdash \mathrm{N} \twoheadrightarrow \mathrm{P}$ is actually a reduction $\lambda^1\beta\eta\pi\vdash \mathrm{N} \twoheadrightarrow \mathrm{P}$, as N contains no $\mathbf{T}$-redex, and no $\mathbf{T}$-redex can be created by the application of reduction rules. The same applies to $\lambda^1\beta\eta\pi*\vdash \mathrm{M} \twoheadrightarrow \mathrm{P}$ and, thus, $\lambda^1\beta\eta\pi\vdash \mathrm{N} = \mathrm{M}$. Similarly for $\lambda^1\beta\eta\pi$ w.r.t $\lambda^1\beta\eta$.

Notation 3.5 Recall that by $\overrightarrow{x}$, $\overrightarrow{y}$, $\overrightarrow{M}$... we denote vectors of variables, terms, etc.

Lemma 3.6 (Terms of $\lambda^1\beta\eta\pi$ whose type is arrow-only belong to $\lambda^1\beta\eta$)
Let M be a term of $\lambda^1\beta\eta\pi$ in normal form such that $M : A$, where A is a type with no occurrence of $\times$ in it. If no free variable of M has a type with occurrences of $\times$, then M is actually a term in $\lambda^1\beta\eta$.

Proof. By induction on the structure of M (see appendix). □

Proposition 3.7 (Isolate the relevant $\langle M_1, \ldots, M_n\rangle$ in an isomorphism)
Let $S \equiv S_1 \times \ldots \times S_m$ and $R \equiv R_1 \times \ldots \times R_n$ be type-n.f.'s where neither the S_i's nor the R_j's contain any occurrences of $\mathbf{T}$ or $\times$. Then $S \cong_p R$ iff there exist $M_1, \ldots, M_n$ and $N_1, \ldots, N_m$ such that

$$x_1 : S_1, \ldots, x_m : S_m \vdash M_1, \ldots, M_n \quad M_i[\overrightarrow{N}/\overrightarrow{x}] =_{\beta\eta} y_i, \text{ for } 1 \le i \le n$$

$$y_1 : R_1, \ldots, y_n : R_n \vdash N_1, \ldots, N_m \quad N_j[\overrightarrow{M}/\overrightarrow{y}] =_{\beta\eta} x_j, \text{ for } 1 \le i \le m$$

(where substitution of vectors of equal length is meant componentwise).

Proof. (See appendix: it is not obvious). □

In conclusion, we have isolated some interesting terms from which every constant has been factored out. Next we prove that provably equal types in normal form have equal length.

Lemma 3.8 (Isomorphic type-n.f.'s have equal length)
Assume that $R_1 \times \ldots \times R_n$ and $S_1 \times \ldots \times S_m$ are type-n.f.'s and $M \equiv \langle M_1, \ldots, M_n\rangle$, $N \equiv \langle N_1, \ldots, N_m\rangle$ are terms in $\lambda^1\beta\eta\pi$ such that

$$x_1 : S_1, \ldots, x_m : S_m \vdash M_1, \ldots, M_n \quad M_i[\overrightarrow{N}/\overrightarrow{x}] =_{\beta\eta} y_i, \text{ for } 1 \le i \le n$$

$$y_1 : R_1, \ldots, y_n : R_n \vdash N_1, \ldots, N_m \quad N_j[\overrightarrow{M}/\overrightarrow{y}] =_{\beta\eta} x_j, \text{ for } 1 \le i \le m$$

then $n = m$ and there exist permutations σ, π over n (and terms P_i, Q_j) such that

$$M_i = \lambda\overrightarrow{u_i}.x_{\sigma_i}\overrightarrow{P}_i \qquad \text{and} \qquad N_j = \lambda\overrightarrow{v_j}.x_{\pi_i}\overrightarrow{Q}_j$$

Proof. By lemma 3.6 (recall that we may assume that each M_i and N_j is in normal form) one has that M_i and N_j are in $\lambda^1\beta\eta$. Then,

$$M_i = \lambda\overrightarrow{u}_i.s_i\overrightarrow{P}_i \qquad \text{and} \qquad N_j = \lambda\overrightarrow{v}_j.t_j\overrightarrow{Q}_j$$

Note that s_i is a free variable (namely some x_j), since $M_i[\overrightarrow{N}/\overrightarrow{x}] =_{\beta\eta} y_i$. Indeed, if s_i is bound then M_i is $\lambda u_1 \ldots s_i \ldots u_k.s_i\overrightarrow{P}_i$ and $M_i[\overrightarrow{N}/\overrightarrow{x}]$ is $\lambda u_1 \ldots s_i \ldots u_k.s_i\overrightarrow{P}_i[\overrightarrow{N}/\overrightarrow{x}]$ so that s_i would still be a bound head variable, and there would be no way to reduce it to a term without abstraction. Similarly t_j is some y_i.

So there are two functions $\sigma : n \to m$, π: $m \to n$ such that

$$M_i = \lambda\overrightarrow{u}_i.x_{\sigma(i)}\overrightarrow{P}_i \text{ for } 1 \le i \le n, \qquad N_j = \lambda\overrightarrow{v}_j.y_{\pi(j)}\overrightarrow{Q}_j \text{ for } 1 \le i \le m$$

In conclusion, for $1 \le i \le n$ we obtain:

$$\begin{aligned}
y_i =_{\beta\eta} M_i[\overrightarrow{N}/\overrightarrow{x}] &=_{\beta\eta} (\lambda\overrightarrow{u_i}.x_{\sigma(i)}\overrightarrow{P}_i)[\overrightarrow{N}/\overrightarrow{x}] \\
&=_{\beta\eta} \lambda\overrightarrow{u_i}.N_{\sigma(i)}\{\overrightarrow{P}_i[\overrightarrow{N}/\overrightarrow{x}]\} \\
&=_{\beta\eta} \lambda\overrightarrow{u_i}.(\lambda\overrightarrow{v}_{\sigma(i)}.y_{\pi(\sigma(i))}\overrightarrow{Q}_{\sigma(i)})\{\overrightarrow{P}_i[\overrightarrow{N}/\overrightarrow{x}]\} \\
&=_{\beta\eta} \text{if } \overrightarrow{v}_{\sigma(i)} \text{ is longer than } \overrightarrow{P}_i \\
&\quad \text{then } \lambda\overrightarrow{u_i}.\overrightarrow{v}_{\sigma(i)}.y_{\pi(\sigma(i))}\overrightarrow{Q}_{\sigma(i)}[(\overrightarrow{P}_i[\overrightarrow{N}/\overrightarrow{x}])/\overrightarrow{u_i}] \\
&\quad \text{else } \lambda\overrightarrow{u_i}.y_{\pi(\sigma(i))}\{\overrightarrow{Q}_{\sigma(i)}[(\overrightarrow{P}_i[\overrightarrow{N}/\overrightarrow{x}])/\overrightarrow{u_i}]\}\{\overrightarrow{\ }
\end{aligned}$$

In either case of the last equality, each term can reduce to y_i iff $y_i = y_{\pi(\sigma(i))}$ and each of the Q's and P's left orderly reduce to one of the bound variables,

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Recall now that all typed terms possess a (unique) normal form (see [Bar84]). As we now need an interplay between typed and type-free terms, we are going to be more explicit about which sort of terms we are dealing with, when needed. Let M be a typed λ-term. We write e(M) for the **erasure** of M, i.e. for M with all type labels on variables erased.

Remark 4.4 *Observe that the erasures of all axioms and rules of the typed lambda calculus are themselves axioms and rules of the untyped lambda calculus. Then, in particular, if M and N are terms of $\lambda^1\beta\eta$ and $\lambda^1\beta\eta \vdash M = N$, one has $\lambda_{\beta\eta} \vdash e(M) = e(N)$.*

Theorem 4.5 *If $M : A \to B$ and $N : B \to A$ are invertible terms in $\lambda^1\beta\eta$, then e(M) and e(N) are f.h.p.'s.*

Proof. e(N M) = e(N)∘e(M), and hence, by the remark, $\lambda_{\beta\eta} \vdash$ e(M)∘e(N) = e(I_σ) = I and $\lambda_{\beta\eta} \vdash$ e(N)∘e(M) = e(I_σ) = I. Thus by Theorem 4.2, e(M) and e(N) are f.h.p.'s. □

The first application of 4.2 we need is the following.

Proposition 4.6 *Let $M_1, \ldots, M_n$ and $N_1, \ldots, N_n$ and permutation σ satisfy all the assumptions in lemma 3.8. Then $\lambda x_{\sigma(i)}.M_i{:}S_{\sigma(i)} \to R_i$ and $\lambda y_i.N_{\sigma(i)}{:}R_i \to S_{\sigma(i)}$ are invertible terms.*

Proof. For a suitable typing of the variables it is possible to build the following terms of $\lambda^1\beta\eta$:

$$M = \lambda z.\lambda x_1 \ldots x_n.zM_1 \ldots M_n, \qquad N = \lambda z.\lambda y_1 \ldots y_n.zN_1 \ldots N_n.$$

It is an easy computation to check, by the definition of the M_i's and of the N_i's, that M and N are invertible. Moreover, they are (by the construction given in the Appendix) in normal form, thus, by Dezani's theorem, (the erasures of) M and N are f.h.p.'s. This is enough to show that every M_i has only one occurrence of the x_i's (namely $x_{\sigma(i)}$); similarly for the N_i's.

Thus we obtain $M_i[\overrightarrow{N}/\overrightarrow{x}] \equiv M_i[N_{\sigma(i)}/x_{\sigma(i)}] =_{\beta\eta} y_i$, for $1 \le i \le n$, and $N_i[\overrightarrow{M}/\overrightarrow{y}] \equiv N_i[M_{\pi(i)}/y_{\pi(i)}] =_{\beta\eta} x_i$, for $1 \le i \le n$,

Hence, for each i, $\lambda x_{\sigma(i)}.M_i{:}S_{\sigma(i)} \to R_i$ and $\lambda y_i.N_{\sigma(i)}{:}R_i \to S_{\sigma(i)}$ are invertible. □

As a result of all the work done so far, we can then focus on invertible terms whose types contain only " → ", i.e. investigate componentwise the isomorphisms of type-n.f.'s. Of course, these isomorphisms will be given just by a fragment of the theory $Th^1_{\times T}$.

Definition 4.7 Let **Swap** be the subtheory of $Th^1_{\times T}$ given by just the following proper axiom (plus the usual axioms and rules for "="),

$$(swap) \qquad A \to B \to C = B \to A \to C.$$

Swap is a subtheory of $Th^1_{\times T}$ by axioms 1 and 3 of $Th^1_{\times T}$.

Proposition 4.8 *Let A, B be type expressions with no occurrences of* **T** *or* $\times$ *. Then $A \cong_p B$ $\Rightarrow$ **Swap** $\vdash A = B$.*

Proof. Suppose A $\cong_p$ B via M and N. As usual, we may assume without loss of generality that M and N are in normal form. By lemma 3.6, M and N actually live in $\lambda^1\beta\eta$ and, by theorem 4.5, e(M) and e(N) are f.h.p.'s. We prove **Swap** $\vdash$ A = B by induction on the depth of the Böhm-tree of M.

Depth 1: M $\equiv \lambda$z : C. z. Thus M : $C \to C$. Now, **Swap** $\vdash$ C = C by reflexivity.

Depth n+1: M $\equiv \lambda$z : E. $\lambda\overrightarrow{x}{:}\overrightarrow{D}$. z$\overrightarrow{N}_\sigma$. Recall z$\overrightarrow{N}_\sigma$ = $(\ldots(\mathrm{zN}_{\sigma(1)})\ldots\mathrm{N}_{\sigma(n)})$ where if the ith abstraction in $\lambda\overrightarrow{x}{:}\overrightarrow{D}$ is $\lambda x_i{:}D_i$ then the erasure of $\lambda x_i{:}D_i.N_i$ is a f.h.p. Let F_i be the type of N_i.

In order to type check, we must have $\mathrm{E} = (\mathrm{F}_{\sigma(1)} \to \ \ldots \to \mathrm{F}_{\sigma(n)} \to B)$ for some B. Thus the type of M is $(\mathrm{F}_{\sigma(1)} \to \ \ldots \to \mathrm{F}_{\sigma(n)} \to B) \to (\mathrm{D}_{\sigma(1)} \to \ \ldots \to \mathrm{D}_{\sigma(n)} \to B)$.

Since $\lambda x_i{:}D_i.N_i$ is a f.h.p, $\lambda x_i{:}D_i.N_i$ gives (half of) a provable isomorphism from $_i$ to F_i. By induction, since the height of the Böhm tree of (of the erasure of) each $\lambda x_i{:}D_i.N_i$ is less than the height of the Böhm tree of M, one has $\mathbf{Swap} \vdash D_i = F_i$ for $1 \leq i \leq n$. By repeated use of the rules for "=", we get

$$\mathbf{Swap} \vdash (\mathrm{F}_{\sigma(1)} \to \ \ldots \to \mathrm{F}_{\sigma(n)} \to B) = (\mathrm{D}_{\sigma(1)} \to \ \ldots \to \mathrm{D}_{\sigma(n)} \to B)$$

Hence it suffices to show

$$\mathbf{Swap} \vdash (\mathrm{D}_{\sigma(1)} \to \ \ldots \to \mathrm{D}_{\sigma(n)} \to B) = (\mathrm{D}_1 \to \ \ldots \to \mathrm{D}_n \to B)$$

This is quite simple to show by repeated use of axiom (swap) above in conjunction with the rules for equality.

□

Clearly, also the converse of proposition 4.8 holds, since the "⇐" part in 4.8 is provable by a fragment of the proof in theorem 2.6. Thus one has:

$$\mathbf{Swap} \vdash A = B \Longleftrightarrow A \cong_p B \text{ by terms in } \lambda^1\beta\eta.$$

The result we aim at is just the extension of this fact to $Th^1_{\times T}$ and $\lambda^1\beta\eta\pi*$.

Theorem 4.9 (Main Theorem) $S \cong_p R \Longleftrightarrow Th^1_{\times T} \vdash S = R$

Proof. In view of theorem 2.6, we only need to prove $\mathrm{S} \cong_p$

Indeed, more can be said about the connection to Category Theory. We also hint here of a simple application to Proof Theory, but refer to [DCL91] for more discussions on both topics.

Take the intuitionistic positive calculus, IPC, i.e. Intuitionistic Logic with only $\rightarrow, \times$ (i.e. conjunction), and True, and consider the following notion of *strong equivalence* (see [Mar92], [LE85] and [AB91]).

Definition 4.11 Two formulas A and B of IPC are *strongly* equivalent iff there are proofs **f** of the sequent A $\vdash$ B and **g** of the sequent B $\vdash$ A such that the proofs **g**$\circ$**f** and **f**$\circ$**g** obtained by composition reduce, by cut-elimination, to the one step deductions A $\vdash$ A and B $\vdash$ B.

Notice that this notion of equivalence is much stronger than the classical notion of logical equivalence: all tautologies of IPC are logically equivalent, for example, but only a few are strongly equivalent.

Corollary 4.12 (Connection with deductive systems) *Two formulas A and B of IPC are strongly equivalent iff* $Th^1_{\times T} \vdash A = B$.

Appendix

We give here the proofs of the lemmas in section 3. The numbers refer to that section.

Lemma 3.1(Form of the terms of a product type) *Given a term M of* $\lambda^1\beta\eta\pi*$ *in normal form such that M:* $A \times B$*, then either* $M \equiv \langle M_1, M_2\rangle$*, for some* M_1*,* M_2*, or there is a free variable* $x : C$ *in M such that* $A \times B$ *is a type subexpression of C.*

Proof. By induction on the length of the structure of M.

Basis of induction: if M is of length 1, then it can be only a free variable of type $A \times B$.

Inductive step: $M \equiv \lambda\overrightarrow{x}.r\,\overrightarrow{P}$, as it is in normal form. Observe first that this case reduces to M $\equiv$ r $\overrightarrow{P}$, as its type is $\alpha \times \beta$, and we proceed by case analysis on r as follows:

r is a variable: then r is free and has type $type(P_1) \rightarrow (... \rightarrow (type(P_n) \rightarrow A \times B)...)$.

r is $\langle M_1, M_2\rangle$

r is a variable: then r has type $type(P_1) \to (... \to (type(P_n) \to C)...)$; by hypothesis, the P_i's are in normal form and in some P_j there are occurrences of a constant $*_A$, so by induction hypothesis there are **T**'s in $type(P_j)$, hence in the type of r. By this, either r is a free variable or (since r occurs among the $\overrightarrow{x}$) there are **T**'s in the type of M.

r is $\langle P, Q\rangle$: then $M \equiv \lambda\overrightarrow{x}.\langle P, Q\rangle$ where P and Q are in normal form. The type of M is $D_1 \to ... \to D_n \to (A \times B)$, with P : A and Q : B, and $*_A$ occurs in P or Q. By inductive hypothesis, either **T** occurs in $A \times B$ (hence in the type of M, too) or in the type of some free variable y of P or Q. In either case, as above, some **T**'s occur in the type of M or in the type of y, which is free in M.

r is $\mathbf{p}_1$ or $\mathbf{p}_2$: then $M \equiv \lambda\overrightarrow{x}.((\mathrm{p}_i M_1)M_2...M_k)$ where:

- M_j is in normal form, for each j.
- $M_1 : S \times U$ with either S or U $\equiv type(M_2) \to (... \to (type(M_k) \to C)...)$.
- $*_A$ occurs in M_j for some j; consider than

  case $j = 1$: then **T** occurs in $S \times U$, by induction hypothesis. By lemma 3.1, as M cannot be a redex, M_1 is not a pair and has a free variable y : C with $S \times U$ a type subexpression of C. Notice that y is also free in $((\mathrm{p}_i M_1)M_2...M_k)$. Thus as in the earlier cases either y is free in M or some **T**'s occur in the type of M (because y is one of the variables in $\overrightarrow{x}$);

  case $j > 1$: then by induction hypothesis either

  (a) there is a **T** occurring in the type of M_j, and, hence, in $S \times U$ or

  (b) there is a free variable y of M_j with type **T** occurring in its type.

  In case (a), we can conclude the proof as in the case for j = 1 above. In case (b), if y is free in M_j then it is also free in $((\mathrm{p}_i M_1)M_2 \ldots M_k)$. We can thus conclude the proof again as for i = 1.

r is $*_A$: then $M \equiv \lambda\overrightarrow{x}.\, *_A M_1$ or $M \equiv \lambda\overrightarrow{x}.*_A$ and the type of M is $D_1 \to ...D_n \to \mathbf{T}$, for some $D_1, \ldots, D_n$.

□

Lemma 3.6(Terms of $\lambda^1\beta\eta\pi$ whose type is arrow-only belong to $\lambda^1\beta\eta$)
Let M be a term of $\lambda^1\beta\eta\pi$

- $M \equiv \lambda\overrightarrow{x}.((\mathrm{p}_i M_1)M_2...M_k)$ implies, by lemma 3.1, that either M_1 is $\langle N_1, N_2\rangle$ or M_1 has a free variable x : C with $S \times U$ a type subexpression of C. The first case is not possible, as $\mathrm{p}_i\langle N_1, N_2\rangle$ is a redex while M is in normal form. Thus M_1 has a free variable x : C with $S \times U$ a type subexpression of C, and, hence, either $x \in FV(M)$ or $S \times U$ is a type subexpression of the type of M, since the type of M includes the types of bound variables. Impossible.

□

Proposition 3.7(Isolate the relevant $\langle M_1, \ldots, M_n\rangle$ in an isomorphism)
Let $S \equiv S_1 \times \ \ldots \times S_m$ and $R \equiv R_1 \times \ \ldots \times R_n$ be type-n.f.'s where neither the S_i's nor the R_j's contain any occurrences of **T** *or* $\times$ *. Then $S \cong_p R$ iff there exist $M_1, \ldots, M_n$ and $N_1, \ldots, N_m$ such that*

$$x_1 : S_1, \ldots, x_m : S_m \vdash M_1, \ldots, M_n \quad M_i[\overrightarrow{N}/\overrightarrow{x}] =_{\beta\eta} y_i, \text{ for } 1 \le i \le n$$

$$y_1 : R_1, \ldots, y_n : R_n \vdash N_1, \ldots, N_m \quad N_j[\overrightarrow{M}/\overrightarrow{y}] =_{\beta\eta} x_j, \text{ for } 1 \le i \le m$$

(where substitution of vectors of equal length is meant componentwise).

Proof. ($\Rightarrow$) Let M°: $S \to R$ and N° : $R \to S$ be closed terms (in normal form) of $\lambda^1\beta\eta\pi*$ such that M°∘N° $= I_R$ and N°∘M° $= I_S$. Then by standard currying, consider the term $\lambda x_1...x_m.M < x_1, ...x_m >$: $(S_1 \to \ldots \to (S_m \to (R_1 \times \ldots \times R_n)...)$, and observe that the normal form M' of $M < x_1, ...x_m >$: $R_1 \times ... \times R_n$, by lemma 3.1, must be of the form $< M_1, ..., M_n >$, with FV(M') $= \{x_1 : S_1, ...x_m : S_m\}$ (by assumption, the S_i's contain no occurrences of $\times$). The same applies for N.

As for the other properties, let

$$M'' \equiv \lambda z.(\lambda x_1...x_m.M^\circ < x_1, ...x_m >)(p_1 z)...(p_m z)$$

and

$$N'' \equiv \lambda z.(\lambda y_1...y_n.N^\circ < y_1, ...y_n >)(p_1 z)...(p_n z),$$

where the x_i's, y_j's, and z are chosen to be distinct.
Then

$$M'' =_\beta \lambda z.M^\circ < p_1 z, ...p_m z >=_\eta \lambda z.M^\circ z =_\eta M^\circ,$$

and similarly

$$N'' =_\beta \lambda z.N^\circ z =_\eta N^\circ.$$

Compute then

$$\begin{aligned}
M^\circ\circ N^\circ \ &=_{\beta\eta} \ M''\circ N'' \equiv \lambda x.(M''(N''x)) \text{ for x a variable not occurring in M" or N".}\\
&=_{\beta\eta} \ \lambda x.(\lambda z.(\lambda x_1...x_m.M')(p_1 z)...(p_m z))(N''x)\\
&=_{\beta\eta} \ \lambda x. < M_1[\overrightarrow{p_j(N''x)}/\overrightarrow{x_j}], ..., M_n[\overrightarrow{p_j(N''x)}/\overrightarrow{x_j}] >\\
&\quad \text{where the substitution is done simultaneously for all } 1 \le j \le m,\\
&=_{\beta\eta} \ \lambda x. < M_1[\overrightarrow{N}[\overrightarrow{p_i x}/\overrightarrow{y}]/\overrightarrow{x}], \ldots, M_n[\overrightarrow{N}[\overrightarrow{p_i x}/\overrightarrow{y}]/\overrightarrow{x}] >\\
&\quad \text{since} \quad N''x \ =_{\beta\eta} \ \lambda y_1...y_n.N'(p_1 x)\ldots(p_n x)\\
&\qquad\qquad =_{\beta\eta} \ < N_1[\overrightarrow{p_i x}/\overrightarrow{y}], ..., N_m[\overrightarrow{p_i x}/\overrightarrow{y}] >\\
&\quad \text{where substitution is done simultaneously for all } 1 \le i \le n,\\
&=_{\beta\eta} \ \lambda x. < M_1[\overrightarrow{N}/\overrightarrow{x}][\overrightarrow{p_i x}/\overrightarrow{y_i}], \ldots, M_n[\overrightarrow{N}/\overrightarrow{x}][\overrightarrow{p_i x}/\overrightarrow{y_i}] >\\
&\quad \text{by substitution properties, as no} y_i \text{ is free in } M'\\
&=_{\beta\eta} \ \lambda x. < p_1 x, ..., p_n x >\\
&\quad \text{since } M^\circ\circ N^\circ =_{\beta\eta} \lambda x.x \text{ and } x =_{\beta\eta} < p_1 x, ..., p_n x > .
\end{aligned}$$

Observe now that the equality just proved implies, componentwise, that $M_k[\overrightarrow{N}/\overrightarrow{x}][\overrightarrow{p_i x}/\overrightarrow{y_i}] =_{\beta\eta} p_k x$. For the purpose of the final argument of the proof, we refer now to $\overset{\beta\eta\pi^*}{\twoheadrightarrow}$ as a "several steps reduction" in $\lambda^1\beta\eta\pi*$. In view of the Church-Rosser property for this calculus, the last equality is equivalent to

$$M_k[\overrightarrow{N}/\overrightarrow{x}][\overrightarrow{p_i w}/\overrightarrow{y_i}] \overset{\beta\eta\pi^*}{\twoheadrightarrow} p_k w.$$

where w is a fresh variable (to avoid confusion between $\overrightarrow{x}$ and x; in other words, w is not free in M_k nor in any N_i and cannot be free in any reduct of $M_k[\overrightarrow{N}/\overrightarrow{x}]$ either.)

Notice now that by hypothesis the terms $\overrightarrow{M}$ and $\overrightarrow{N}$ are in normal form and have no **T** or $\times$ involved in their types or in the types of their free variables (the $\overrightarrow{S_i}$ and $\overrightarrow{R_i}$), so by lemma 3.6 they are actually terms of $\lambda^1\beta\eta$. This allows us to conclude that the substitution $[\overrightarrow{p_i w}/\overrightarrow{y_i}]$ creates no new redexes: the $\overrightarrow{p_i w}$ could only create new redexes for surjective pairing reductions, i.e. when they appear in $< p_1 w, ..., p_n w >$. But VecM and $\overrightarrow{N}$ do not contain any pair, so surjective pairing reductions cannot apply.

This fact has an important consequence: the reductions are actually performed inside $M_k[\overrightarrow{N}/\overrightarrow{x}]$, so if we have $M_k[\overrightarrow{N}/\overrightarrow{x}][\overrightarrow{p_i w}/\overrightarrow{y_i}] \overset{\beta\eta\pi^*}{\twoheadrightarrow} Q$, then $M_k[\overrightarrow{N}/\overrightarrow{x}] \overset{\beta\eta\pi^*}{\twoheadrightarrow} Q'$ with $Q \equiv Q'[\overrightarrow{p_i w}/\overrightarrow{y_i}]$.

This implies, in the case of $M_k[\overrightarrow{N}/\overrightarrow{x}][\overrightarrow{p_i w}/\overrightarrow{y_i}] \overset{\beta\eta\pi^*}{\twoheadrightarrow} p_k w$, the reduction $M_k[\overrightarrow{N}/\overrightarrow{x}] \overset{\beta\eta\pi^*}{\twoheadrightarrow} Q'$ with $p_k w \equiv Q'[\overrightarrow{p_i w}/\overrightarrow{y_i}]$, that is $M_k[\overrightarrow{N}/\overrightarrow{x}] \overset{\beta\eta\pi^*}{\twoheadrightarrow} p_k w$. In conclusion, $M_k[\overrightarrow{N}/\overrightarrow{x}] = y_k$, as required.

Symmetrically, one obtains $N_j[\overrightarrow{M}/\overrightarrow{y}] =_{\beta\eta} x_j$ from $\mathrm{N}^\circ \circ \mathrm{M}^\circ = \lambda \mathrm{x.x}$.

($\Leftarrow$) Just step through the above proof in reverse order, defining the required closed terms by

$$M \equiv \lambda z.(\lambda x_1 ... x_m. < M_1, ... M_n >)(p_1 z)...(p_m z),$$

$$N \equiv \lambda z.(\lambda y_1 ... y_n. < N_1, ... N_m >)(p_1 z)...(p_n z).$$

□

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